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Continued fractions and q-series generating functions for the generalized sum-of-divisors functions

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We construct new continued fraction expansions of Jacobi-type J-fractions in z whose power series expansions generate the ratio of the q-Pochhammer symbols, $(a;q)_n/(b;q)_n$, for all integers $n \geq 0$ and where $a, b, q \in \mathbb{C}$ are non-zero and defined such that $|q| < 1$ and $|b/a| < |z| < 1$. If we set the parameters $(a, b) := (q, q^2)$ in these generalized series expansions, then we have a corresponding J-fraction enumerating the sequence of terms $(1-q)/(1-q^{n+1})$ over all integers $n \geq 0$. Thus we are able to define new q-series expansions which correspond to the Lambert series generating the divisor function, $d(n)$, when we set $z \mapsto q$ in our new J-fraction expansions. By repeated differentiation with respect to z, we also use these generating functions to formulate new q-series expansions of the generating functions for the sums-of-divisors functions, $\sigma_\alpha(n)$, when $\alpha \in \mathbb{Z}^+$. To expand the new q-series generating functions for these special arithmetic functions we define a generalized class of so-termed Stirling-number-like "q-coefficients", or Stirling q-coefficients, whose properties, relations to elementary symmetric polynomials, and relations to the convergents to our infinite J-fractions are also explored within the results proved in the article.

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1. Introduction

1.1. Continued fraction expansions of ordinary generating functions

Expansions of Jacobi-type J-fractions. *Jacobi-type continued fractions*, or *J-fractions*, correspond to power series defined by infinite continued fraction expansions of the form[1]

$$J_\infty(z) = \cfrac{1}{1 - c_1 z - \cfrac{\mathrm{ab}_2\, z^2}{1 - c_2 z - \cfrac{\mathrm{ab}_3\, z^2}{\cdots}}} \tag{1}$$

$$= 1 + c_1 z + \left(\mathrm{ab}_2 + c_1^2\right) z^2 + \left(2\,\mathrm{ab}_2\, c_1 + c_1^3 + \mathrm{ab}_2\, c_2\right) z^3 + \left(\mathrm{ab}_2^2 + \mathrm{ab}_2\,\mathrm{ab}_3 + 3\,\mathrm{ab}_2\, c_1^2 + c_1^4 + 2\,\mathrm{ab}_2\, c_1 c_2 + \mathrm{ab}_2\, c_2^2\right) z^4 + \cdots, \tag{2}$$

for arbitrary, application-specific implicit sequences $\{c_i\}_{i=1}^{\infty}$ and $\{\mathrm{ab}_i\}_{i=2}^{\infty}$, and some typically formal series variable $z \in \mathbb{C}$ [14, *cf.* §3.10], [19]. The formal series enumerated by special cases of the truncated and infinite J-fraction series of this form include typically divergent *ordinary* (as opposed to typically closed-form *exponential*) *generating functions* for many one and two-index combinatorial sequences including the so-termed "*square series*" functions studied in the references and in the results from Flajolet's articles [7,8,16].

Generalized properties of the convergents to infinite J-fractions. We define the h-th convergent functions, $\mathrm{Conv}_h(z) := \mathrm{P}_h(z)/\,\mathrm{Q}_h(z)$, to the infinite J-fraction in (1) recursively through the component numerator and denominator functions given by[2]

$$\begin{aligned} \mathrm{P}_h(z) &= (1 - c_h \cdot z)\,\mathrm{P}_{h-1}(q,z) - \mathrm{ab}_h \cdot z^2\,\mathrm{P}_{h-2}(q,z) + [h=1]_\delta \\ \mathrm{Q}_h(z) &= (1 - c_h \cdot z)\,\mathrm{Q}_{h-1}(q,z) - \mathrm{ab}_h \cdot z^2\,\mathrm{Q}_{h-2}(q,z) + (1 - c_1 \cdot z)\,[h=1]_\delta + [h=0]_\delta. \end{aligned} \tag{3}$$

If we let $j_n := [z^n]J_\infty(z)$ in (1), the convergents to the full J-fraction defined as above provide $2h$-order accurate truncated power series approximations to the infinite-order J-fraction generating functions in the following form for each $h \geq 1$:

$$\mathrm{Conv}_h(z) = j_0 + j_1 z + j_2 z^2 + \cdots + j_{2h-1} z^{2h-1} + \sum_{n \geq h} \bar{j}_{h,n} z^n.$$

[1] Conventions: We adopt a hybrid of the notation for the implicit continued fraction sequences $a_{h-1}b_h :\mapsto \mathrm{ab}_h$ from Flajolet's article [7]. Our usage of P/Q to denote the convergent function ratios is also consistent with the conventions from this reference.

[2] Special notation: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n=k]_\delta \equiv \delta_{n,k}$. Similarly, $[\mathtt{cond} = \mathtt{True}]_\delta \equiv \delta_{\mathtt{cond},\mathtt{True}} \in \{0,1\}$, which is 1 if and only if `cond` is true, in the remainder of the article.

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