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Abstract

We give an extension of a theorem of Cesàro from the rational integers to the ring of integers of an arbitrary number field. This extension is used to generalize Pillai's function to number fields.

Keywords: Dedekind zeta function, Euler totient function, Gcd-sum function, Number field, Pillai function
2010 MSC: 11R42, 11R99

1. Introduction

A theorem of Cesàro (see e.g. [5], [8, p.127], [17]) states that for every natural number $n \in \mathbb{N} = \{1, 2, \ldots\}$ and any arithmetical function f we have

$$\sum_{i=1}^{n} f((i,n)) = \sum_{d|n} f(d)\varphi(n/d), \tag{1}$$

where (i, n) denotes the greatest common divisor of i and n, and φ is the Euler totient function. There is in the literature a large number of generalizations and analogues of Cesàro theorem. For a rich and extensive survey concerning generalizations of Cesàro theorem the reader is referred to [10]. Since rings of integers in a number field are natural generalizations of the rational integers, the question natural arises as to whether an analogous statement could be made for the ring of integers in a number field. Recently, there has been considerable interest in extending arithmetical identities from the rational integers to a more general setting (see e.g. [11], [13]).

The aim of this paper is to extend Cesàro theorem to the ring O_K of integers in a number field K. Instead of working with elements in O_K, where unique factorization can fail, we will work with ideals. As is well known using ideals in place of elements we can save unique factorization. More precisely, every nonzero ideal $\mathfrak{n}$ of O_K can be uniquely written in the form $\mathfrak{n} = \mathfrak{p}_1^{\alpha_1} \ldots \mathfrak{p}_s^{\alpha_s}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are distinct nonzero prime ideals and $\alpha_1, \ldots, \alpha_s$ are positive rational integers (see e.g. [14, p.8]).

Unique factorization of ideals in O_K permits calculations that are analogous to some familiar manipulations involving ordinary integers. In particular we can define the concept of arithmetical function on the set of ideals of O_K. A real or complex-valued function defined on the set of ideals of the ring of integers in a number field is called an arithmetical function. As a very simple example, consider a nonzero ideal $\mathfrak{n}$ of O_K, then the generalized Euler totient function, which is denoted by $\varphi_K(\mathfrak{n})$, is defined to be the order of the multiplicative group of units in the factor ring $O_K/\mathfrak{n}$, denoted by $U(O_K/\mathfrak{n})$, with the convention that $\varphi_K(O_K) = 1$. That is,

$$\varphi_K(\mathfrak{n}) = \begin{cases} 1 & if \;\; \mathfrak{n} = O_K, \\ |U(O_K/\mathfrak{n})| & otherwise. \end{cases}$$