ARTICLE IN PRESS YJNTH:5777

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

The number of representations of squares by integral ternary quadratic forms ☆

Kyoungmin Kim [a], Byeong-Kweon Oh [b,*]

[a] *Department of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea*
[b] *Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea*

ARTICLE INFO

Article history:
Received 30 September 2015
Received in revised form 22 May 2017
Accepted 22 May 2017
Available online xxxx
Communicated by S.J. Miller

MSC:
primary 11E12, 11E20

Keywords:
Representations of ternary quadratic forms
Squares

ABSTRACT

Let f be a positive definite integral ternary quadratic form and let $r(k,f)$ be the number of representations of an integer k by f. In this article we study the number of representations of squares by f. We say the genus of f, denoted by $\mathrm{gen}(f)$, is indistinguishable by squares if for every integer n, $r(n^2,f)=r(n^2,f')$ for every quadratic form $f'\in\mathrm{gen}(f)$. We find some non-trivial genera of ternary quadratic forms which are indistinguishable by squares. We also give some relation between indistinguishable genera by squares and the conjecture given by Cooper and Lam, and we resolve their conjecture completely.

1. Introduction

For a positive definite integral ternary quadratic form

☆ This work was supported by the National Research Foundation of Korea (NRF-2014R1A1A2056296) and (NRF-2017R1A2B4003758).

* Corresponding author.
E-mail addresses: kiny30@snu.ac.kr (K. Kim), bkoh@snu.ac.kr (B.-K. Oh).

http://dx.doi.org/10.1016/j.jnt.2017.05.012
0022-314X/© 2017 Elsevier Inc. All rights reserved.

Please cite this article in press as: K. Kim, B.-K. Oh, The number of representations of squares by integral ternary quadratic forms, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.05.012

$$f(x_1,x_2,x_3)=\sum_{1\leqslant i,j\leqslant 3} a_{ij}x_ix_j \qquad (a_{ij}=a_{ji}\in\mathbb{Z})$$

and an integer n, we define a set $R(n,f)=\{(x_1,x_2,x_3)\in\mathbb{Z}^3 : f(x_1,x_2,x_3)=n\}$, and $r(n,f)=|R(n,f)|$. It is well known that $r(n,f)$ is always finite if f is positive definite. Finding a closed formula for $r(n,f)$ or finding all integers n such that $r(n,f)\neq 0$ for an arbitrary ternary quadratic form f are quite old problems which are still widely open. As one of the simplest cases, Gauss showed that if f is a sum of three squares, then $r(n,f)$ is a multiple of the Hurwitz–Kronecker class number. In fact, if the class number of f is one, that is, every quadratic form which is locally isometric to f is isometric to it globally, then Minkowski–Siegel formula gives a closed formula for $r(n,f)$. As a natural modification of the Minkowski–Siegel formula, it was proved in [13] and [17] that the weighted sum of the representations of quadratic forms in the spinor genus is also equal to the product of local densities except spinor exceptional integers (see also [16] for spinor exceptional integers). Hence if the spinor class number $g^+(f)$ of f is one, we also have a closed formula for $r(n,f)$. As far as the authors know, there is no known closed formula for $r(n,f)$ except those cases (for some relations between $r(n,f)$'s, see [10]). If $r(n,f)=r(n,f')$ for every $f'\in\text{gen}(f)$, it is certain that Minkowski–Siegel formula gives a closed formula for $r(n,f)$. However, Schiemann proved in [15] that for two positive definite ternary forms f and f', if $r(n,f)=r(n,f')$ for every positive integer n, then f is isometric to f'. Here $\text{gen}(f)$ denotes the genus of f, which is set of all positive definite ternary quadratic forms that are isometric to f over the p-adic integer ring $\mathbb{Z}_p$ for every prime p.

If we consider a proper subset S of positive integers, then it might be possible that there are non-isometric forms f and f' such that $r(n,f)=r(n,f')$ for every integer $n\in S$. In this article, we consider the case when S is the set of perfect squares. We say the genus of a ternary quadratic form f is *indistinguishable by squares* if $r(n^2,f)=r(n^2,f')$ for every $f'\in\text{gen}(f)$ and every integer n. It is obvious that if the genus of f does not represent any squares of integers, that is, $r(n^2,f')=0$ for every integer n and every $f'\in\text{gen}(f)$, or the class number of f is one, then the genus of f is indistinguishable by squares. If the genus of f is indistinguishable by squares, then Minkowski–Siegel formula gives a closed formula for $r(n^2,f)$ for every integer n.

In 2013, Cooper and Lam gave a conjecture in [5] on the representations of squares by diagonal ternary quadratic forms representing 1. In that article, they proved by using some q-series identities, that for the quadratic form $f=x^2+by^2+cz^2$ with $(b,c)=(1,1),(1,2),(1,3),(2,2),(3,3)$,

$$r(n^2,f)=\prod_{p\mid 2bc} g(b,c,p,\text{ord}_p(n))\prod_{p\nmid 2bc} h(b,c,p,\text{ord}_p(n)),$$

where

$$h(b,c,p,\text{ord}_p(n))=\frac{p^{\text{ord}_p(n)+1}-1}{p-1}-\left(\frac{-df}{p}\right)\frac{p^{\text{ord}_p(n)}-1}{p-1}$$