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TWIN PRIMES AND THE PARITY PROBLEM

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ABSTRACT. We formulate a conjecture regarding the equidistribution of the Möbius function over shifted primes in arithmetic progressions. Our main result is that such a conjecture for a fixed even integer h, in conjunction with the Elliott-Halberstam conjecture, can resolve the parity barrier and produce infinitely many primes p such that $p+h$ is also prime.

1. INTRODUCTION

Let $\Lambda(n)$ denote the von Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p, \\ 0 & \text{otherwise.} \end{cases}$$

Let h be a fixed even integer. It is conjectured that

$$\sum_{n\leq x} \Lambda(n)\Lambda(n+h) \sim \mathfrak{S}(h)x, \tag{1.1}$$

where $\mathfrak{S}(h)$ is the singular series defined as

$$\mathfrak{S}(h) = \prod_{p|h}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid h}\left(1-\frac{1}{(p-1)^2}\right).$$

If $h = 2$, this gives an asymptotic formula for the number of twin primes. It is believed that sieve methods cannot resolve this conjecture because of the parity problem.

The term "parity principle" was first coined by Atle Selberg [24], who came across this phenomenon in 1946, in his work on the ingenious sieve that bears his name today. He described this principle as follows: *"Sets of integers tend to be very evenly distributed with respect to the parity of their number of prime factors unless they have been particularly produced, constructed or selected in a way that has a built in bias."* The discussion by Selberg in [25] indicates that sieve methods are unable to distinguish whether an integer has an odd or an even number of prime factors. This is commonly referred to as the parity problem.

The parity problem can also be explained in the context of Bombieri's asymptotic sieve [2], which highlights that classical sieve methods are unable to sift out numbers having exactly r prime factors, irrespective of the choice of r. There have been a number of attempts (c.f. [14], [6], [9], [11]) in various settings, to break the parity barrier by postulating additional analytic data into the sieve machinery. In this article, we follow this line of thought. A related result is due to J. Friedlander and H. Iwaniec [10] who assumed an estimate for certain bilinear forms, relying upon cancellations arising from sign changes of the Möbius function, in order to circumvent the parity problem and show the infinitude of primes of the form a^2+b^4.

A problem closely related to the parity principle is that of showing significant cancellation in the summatory function of the Möbius function:

$$M(x) := \sum_{n\leq x}\mu(n).$$

The assertions $M(x) = o(x)$ and $M(x) \ll_\epsilon x^{\frac{1}{2}+\epsilon}$ are equivalent to the prime number theorem and the Riemann hypothesis respectively. A higher rank version of this was conjectured by S.

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