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The values of cubic forms at prime arguments

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A R T I C L E I N F O

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A B S T R A C T

Let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative. Let $\mathcal{V}$ be a well-spaced sequence, $\delta > 0$. If λ_1/λ_2 is irrational and algebraic, then we prove that $E(\mathcal{V}, X, \delta) \ll X^{17/18+2\delta+\varepsilon}$, where $E(\mathcal{V}, X, \delta)$ denotes the number of $v \in \mathcal{V}$ with $v \leq X$ such that the inequality $|\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^3 - v| < v^{-\delta}$ has no solution in primes p_1, p_2, p_3, p_4, p_5. Further, we assume that except for one, all other the ratios λ_k/λ_l $(1 \leq k < l \leq 5)$ are irrational and algebraic, then 17/18 can be replaced by 11/12. These improve the earlier results.

1. Introduction

A formal application of the Hardy–Littlewood method suggests that whenever s and k are natural numbers with $s \geq k + 1$, then all large integers n satisfying appropriate local conditions should be represented as the sum of s kth powers of prime numbers. We write

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$$\mathcal{N}_5 = \{n \in \mathbb{N} : n \equiv 1(\text{mod}2),\ n \not\equiv 0, \pm 2(\text{mod}9),\ n \not\equiv 0(\text{mod}7)\},$$

and

$$E_5(N) = |\{1 \le n \le N : n \in \mathcal{N}_5 \text{ and } n \notin \mathcal{A}_5\}|,$$

where

$$\mathcal{A}_5 = \{p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 : p_1, p_2, p_3, p_4, p_5 \text{ are prime numbers}\}.$$

Hua [6] established that almost all numbers in $\mathcal{N}_5$ can be represented as sums of five cubes of prime numbers. Precisely, Hua proved that $E_5(N) \ll N \log^{-A} N$ for some positive number A. There have been also a series of recent advances (see [9,8,11,16,17]).

Davenport and Heilbronn first considered the Diophantine inequalities. Given $k \ge 1$ and s nonzero real numbers $\lambda_1, \cdots, \lambda_s$ (not all in rational ratio, not all negative), we write

$$F(\mathbf{p}) = \sum_{j=1}^{s} \lambda_j p_j^k,$$

where $\mathbf{p} = (p_1, \cdots, p_s)$ with each p_j a prime. Various authors have considered the distribution of values of such forms, for example, see [14]. Here we continue in the direction started by Brüdern, Cook and Perelli [1] and followed by Cook and Fox [3], Cook [2], Harman [5] and Cook and Harman [4]. We call a set of positive reals $\mathcal{V}$ a well-spaced set if there is a $c > 0$ such that

$$u, v \in \mathcal{V}, \quad u \ne v \quad \Rightarrow |u - v| > c.$$

We further assume that

$$|\{v \in \mathcal{V} : 0 \le v \le X\}| \gg X^{1-\varepsilon}.$$

In this paper, let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers, not all negative, let $\mathcal{V}$ be a well-spaced sequence, and let $E(\mathcal{V}, X, \delta)$ denote the number of $v \in \mathcal{V}$ with $v \le X$ such that the inequality

$$|\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^3 + \lambda_5 p_5^3 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4, p_5.

In [4], Cook and Harman show that if λ_1/λ_2 is irrational and algebraic, then one has

$$E(\mathcal{V}, X, \delta) \ll X^{1-\frac{2}{3}\rho(3)+2\delta+\varepsilon} \tag{1.1}$$

for any $\varepsilon > 0$, where $\rho(3) = \frac{1}{14}$, since they use bounds for the exponential sums which arise [10].