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On the average number of divisors of reducible quadratic polynomials

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ABSTRACT

We give an asymptotic formula for the divisor sum $\sum_{c<n\le N} \tau\left((n-b)(n-c)\right)$ for integers $b < c$ of the same parity. Interestingly, the coefficient of the main term does not depend on the discriminant as long as it is a full square. We also provide effective upper bounds of the average divisor sum for some of the reducible quadratic polynomials considered before, with the same main term as in the asymptotic formula.

1. Introduction

Let $\tau(n)$ denote the number of positive divisors of the integer n and $P(x) \in \mathbb{Z}[x]$ be a polynomial. There are many results on estimating average sums of divisors

$$\sum_{n=1}^{N} \tau\left(P(n)\right), \tag{1}$$

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one of which was obtained by Erdős [8], who showed that for an irreducible polynomial $P(x)$ and for any $N > 1$, we have

$$N \log N \ll_P \sum_{n=1}^{N} \tau(P(n)) \ll_P N \log N \, .$$

Here the implied constants can depend both on the degree and the coefficients of the polynomial. When $P(x)$ is a quadratic polynomial Hooley [14] and McKee [18,19] obtained asymptotic formulae for the sum (1). When $\deg P(x) \geq 3$ no asymptotic formulae for (1) are known. A certain progress in this direction was made by Elsholtz and Tao in §7 of [7].

When the polynomial $P(x)$ is *reducible* the behavior is a little bit different. Ingham [15] considered the additive divisor problem and proved that for a fixed positive integer q the following asymptotic holds

$$\sum_{n \leq N} \tau(n)\tau(n+q) \sim \frac{6}{\pi^2}\sigma_{-1}(q) N \log^2 N \, ,$$

as $N \to \infty$, where $\sigma_a(q) = \sum_{d|q} d^a$ for $a, q \in \mathbb{Z}$. Later Hooley [14] predicted that

$$\sum_{n \leq N} \tau(n^2 - r^2) = A(r) N \log^2 N + \mathcal{O}(N \log N) \, , \tag{2}$$

but only recently Dudek [5] provided the exact value of the constant $A(1)$, namely $1/\zeta(2) = 6/\pi^2$. The first aim of this paper is to extend Dudek's work and to find the exact values of $A(r)$ for any integer $r \geq 1$. Actually we find the main term in the asymptotic formula for (1) for slightly more general polynomials $P(n) = (n-b)(n-c)$ for integers $b < c$, such that $b + c$ is even.

For integers $k \geq 0$ and $d > 0$ we define

$$\rho_k(d) := \# \left\{ 0 \leq x < d : \quad x^2 \equiv k \pmod{d} \right\} . \tag{3}$$

The main result we need for the asymptotic estimate of the average divisor sum (1) for reducible quadratic polynomials $P(x)$ is the following Theorem, which is of interest of its own.

Theorem 1. *For any integer $r \geq 1$ we have the asymptotic formula*

$$\sum_{\lambda \leq N} \rho_{r^2}(\lambda) \sim \frac{6}{\pi^2} N \log N,$$

as $N \to \infty$.

From Theorem 1 we can deduce our asymptotic result.

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