Journal of Number Theory 180 (2017) 756–768

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On integer sequences in product sets

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ARTICLE INFO

Article history:
Received 3 November 2015
Received in revised form 15 April 2017
Accepted 7 June 2017
Available online 17 July 2017
Communicated by S.J. Miller

Keywords:
Product sets
Integer sequences in product sets
Number of consecutive terms of a polynomial
Fibonacci numbers in a product set

ABSTRACT

Let B be a finite set of natural numbers or complex numbers. Product set corresponding to B is defined by $B.B := \{ab : a, b \in B\}$. In this paper we give an upper bound for longest length of consecutive terms of a polynomial sequence present in a product set accurate up to a positive constant. We give a sharp bound on the maximum number of Fibonacci numbers present in a product set when B is a set of natural numbers and a bound which is accurate up to a positive constant when B is a set of complex numbers.

1. Introduction

In [5] and [4] Zhelezov has proved that if B is a set of natural numbers then the product set corresponding to B cannot contain long arithmetic progressions. In [5] it was shown that the longest length of arithmetic progression is at most $O(|B| \log |B|)$. We try to generalize this result for polynomial sequences. Let $P(x) \in \mathbb{Z}[x]$ be a non-constant polynomial with positive leading coefficient. Let R be the longest length of consecutive terms of the polynomial sequence contained in the product set $B.B$, that is,

$$R = max\{n : \text{there exists an } x \in \mathbb{N} \text{ such that } \{P(x+1), \cdots, P(x+n)\} \subset B.B\}.$$

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http://dx.doi.org/10.1016/j.jnt.2017.06.001

We prove that R cannot be large for a given non-constant polynomial $P(x)$. In section 3 we consider the question of determining maximum number of Fibonacci and Lucas sequence terms in a product set.

Let $A \times B$ denote the cartesian product of sets A and B. As in [5] we define an auxiliary bipartite graph $G(A, B.B)$ and auxiliary graph $G'(A, B.B)$ which are constructed for any sets A and B whenever $A \subset B.B$. The vertex set of $G(A, B.B)$ is a union of two isomorphic copies of B namely $B_1 = B \times \{1\}$ and $B_2 = B \times \{2\}$ and vertex set of $G'(A, B.B)$ is one isomorphic copy of B namely $B_1 = B \times \{1\}$. For each $a \in A$ we pick a unique representation $a = b_1 b_2$ where $b_1, b_2 \in B$ and place an edge joining $(b_1, 1)$, $(b_2, 2)$ in $G(A, B.B)$ and place an edge joining $(b_1, 1)$, $(b_2, 1)$ in $G'(A, B.B)$.

Note that the number of vertices in $G(A, B.B)$ is $2|B|$ where as number of vertices in $G'(A, B.B)$ is $|B|$. Number of edges in both $G(A, B.B)$ and $G'(A, B.B)$ is $|A|$. Observe that $G'(A, B.B)$ can have self loops and $G(A, B.B)$ cannot have self loops and that $G(A, B.B)$ is necessarily a bipartite graph where as $G'(A, B.B)$ may not be a bipartite graph.

2. Polynomial sequences

Given a non-constant polynomial $P(x)$ with positive leading coefficient and integer coefficients. Since there can be at most finitely many natural numbers r such that $P(r) \leq 0$ or $P'(r) \leq 0$ there exists an l such that $P(r + l) > 0$ and $P'(r + l) > 0$ for all $r \geq 1$. Hence we can assume without loss of generality that every irreducible factor $g(x)$ of $P(x)$ we have $g(x) > 0$ and $g'(x) > 0 \;\; \forall x \geq 1$, as this assumption only effects R by a constant. From now on we will be assuming that for every irreducible divisor $g(x)$ of $P(x)$, $g(x) > 0$ and $g'(x) > 0$ for all natural numbers x. We prove three lemmas in order to obtain an upper bound on R.

From now we let $f(x) \in \mathbb{Z}[x]$ denote an irreducible polynomial divisor of $P(x)$. If $f(x)$ is a polynomial of degree ≥ 2. Let D be the discriminant of $f(x)$. Let d be the greatest common divisor of the set $\{f(n) : n \in \mathbb{N}\}$. Let $f_1(x) = \frac{f(x)}{d}$. Denote $|D|d^2$ by M. If p is a prime divisor of M such that $p^e \| M$, that is $p^e | M$ and $p^{e+1} \nmid M$, then $p^e \nmid d$ and hence there exists an a_p, such that $f_1(x)$ is not divisible by p for all $x \equiv a_p (\mod p^e)$. From Chinese remainder theorem there exists an integer a such that $a \equiv a_p (\mod p^e)$ for all primes p dividing M and hence there exists an a such that $f_1(x)$ is relatively prime to M for all $x \equiv a (\mod M)$.

Lemma 2.1. *For sufficiently large R the number of numbers in the set $\{f_1(r + i) : 1 \leq i \leq R, r + i \equiv a \mod M\}$ with at least one prime factor greater than R is $\geq \frac{R}{3M}$ for every non-negative integer r.*

Proof. Let

$$Q = \prod_{\substack{i=1 \\ r+i \equiv a \mod M}}^{R} f_1(r + i).$$