The Euler-Riemann zeta function in some series formulae and its values at odd integer points

Milorad R. Stevanović, Predrag B. Petrović

PII: S0022-314X(17)30218-4
DOI: http://dx.doi.org/10.1016/j.jnt.2017.05.016
Reference: YJNTH 5781

To appear in: *Journal of Number Theory*

Received date: 30 November 2016
Revised date: 8 May 2017
Accepted date: 19 May 2017

The Euler-Riemann zeta function in some series formulae and its values at odd integer points

Milorad R. Stevanović, Predrag B. Petrović

University of Kragujevac, Faculty of Technical Sciences Čačak
Svetog Save 65, 32000 Čačak, SERBIA
E-mail: predrag.petrovic@ftn.kg.ac.rs

Abstract: The paper presents formulae for certain series involving the Riemann zeta function. These formulae are generalizations, in a natural way, of well known formulae, originating from Leonhard Euler. Formulaes that existed only for initial values n=0, 1 are now found for every natural n. Relevant connections with various known results are also pointed out.
MSC: primary 11M06; secondary 33E99, 33B15
Keywords: Riemann zeta function, odd integer, Raabe's integral, Abel's criteria, Bernoulli numbers.

1. Introduction

The Riemann zeta function $\zeta(s)$ plays a central role in the applications of complex analysis related to number theory (e.g. the generation of irrational and prime numbers) and also as an important tool in signal analysis in many fields. Historically [11], people prefer studying the closed form of the Riemann zeta function at positive integer arguments in that those special values seem to dictate the properties of the objects they are associated with. In condensed matter physics for instance, the famous Sommerfeld expansion, which is useful for calculation of particle number and internal energy of electrons, involves Riemann zeta function at even integers [1], while the spin-spin correlation functions of isotropic spin-1/2 Heisenberg model are expressed by ln2 and Riemann zeta functions with odd integer arguments [12].

The calculation of Riemann zeta function and related series is a hot topic in computational mathematics [4]. The traditional methods are Euler-Maclaurin formula and Riemann-Siegel formula, and algorithms are still being developed in earnest ever since [2, 3, 9, 14]. Typically, a particular numerical method is limited to a special domain. Therefore, when concentrating on Riemann zeta function at odd integers, a special method ought to be constructed in view of the connection of Riemann zeta function values between odd and even integers.

The main goal of the paper is to prove generalization of certain summation formulas involving special values of the Riemann zeta function at even and at odd integers. In [13], Srivastava presents various types of summation formulae related to the Riemann zeta function. This line of investigation has its origin in Euler's memoirs from 1769 and 1781 (see Glaisher [6], p.27, 28). In Johnson's paper [7], from 1905 -1906, the following formula can be found ([13], 2.32)

$$L_n = \sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k+n} = \log 2 \;\; for\ n=0\,. \tag{1.1}$$

Wilton's paper [16], from 1922-1923, introduces the formula ([13], 2.35)