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On the special linear group over orders in quaternion division algebras ☆

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ABSTRACT

It is the aim of this article to give a detailed calculation of the number of cusps of an arithmetic subgroup of $SL_2(D)$ where D is a central simple division algebra over an algebraic number field. We show how this number of cusps corresponds to the class number of D.

1. Introduction

Let k be an algebraic number field and let $\mathcal{O}_k$ be its ring of integers. Consider the special linear group of (2×2)-matrices SL_2/k viewed as an algebraic group defined over k.

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The set of real points of the $\mathbb{Q}$-group $Res_{k/\mathbb{Q}}SL_2$ obtained by restriction of scalars from k to $\mathbb{Q}$ will be denoted by G_∞. The symmetric space X which corresponds to this real semisimple Lie group is a direct product of finitely many copies of the upper half plane H_2 and the three dimensional hyperbolic space H_3. Any arithmetic subgroup Γ of $SL_2(k)$ acts on X, and this action is properly discontinuous. If Γ is torsion free the quotient space X/Γ is a complete Riemannian manifold, non-compact but of finite volume.

In the case $k = \mathbb{Q}$, it is a classical result that H_2/Γ for an arithmetic subgroup $\Gamma \subset SL_2(\mathbb{Z})$, can be compactified by adding finitely many points, the so-called cusps. This might have been known as early as 1880 by Poincaré, though he did not mention it explicitly (see [BJ06] and [Leh64, Ch. 1]).

This construction was extended to the general case of an arbitrary algebraic number field to arithmetic subgroups $\Gamma \subset SL_2(\mathcal{O}_k)$. For example Siegel shows as a decisive result in [Sie61, Prop. 20] that the number of cusps which are added to compactify $X/SL_2(\mathcal{O}_k)$ equals h_k, the class number of k.

In terms of the underlying algebraic group SL_2/k this number of cusps can be interpreted as the number of Γ-conjugacy classes of proper parabolic k-subgroups of $SL_2(k)$.

The situation changes quite a bit and presents interesting phenomena if one considers the special linear group of (2×2)-matrices with entries in a finite-dimensional central simple division algebra D over k and arithmetic subgroups originating from maximal orders in D. More precisely, the algebraic k-group $G = SL_2(D)/k$ is simple, simply-connected and of k-rank equal to 1. Its set of k-points will be denoted by G_k. For every maximal order Λ in D, $G_\Lambda := SL_2(\Lambda)$ is an arithmetic subgroup of G_k.

The number of cusps $cs(G_\Lambda)$ of the subgroup G_Λ is defined as the cardinality of the double quotient $G_\Lambda\backslash G_k/P_k$, where P is a (any) proper parabolic k-subgroup of G. This double quotient, respectively its cardinality, is the main subject of this article.[1] In [Bor63, Prop. 7.5] Borel shows quite generally that $cs(G_\Lambda)$ equals the class number of the algebraic group P with respect to the lattice induced by Λ.[2]

The calculation of the number of cusps of a maximal order Λ in D is split into two cases, depending on the arithmetic of D:

- D is totally definite, i.e. $D_v \cong \mathbb{H}$ the Hamiltonian quaternions for all infinite places v of k.
- D is not totally definite.

If D is not totally definite, one obtains that $cs(G_\Lambda) = h_D$, where h_D is the number of isomorphism classes of left Λ-ideals in D for any maximal order Λ in D. In particular, it is independent of the choice of maximal order Λ in D. This result was obtained by

[1] Note that the number of cusps equals the number of boundary components in the Borel–Serre compactification, and is therefore important in the calculation of the cohomology groups of Λ.

[2] A thorough study of class numbers can be found in [PR94, Chapter 5 and 8].