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Values of pairs involving one quadratic form and one linear form at S-integral points

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Abstract

We prove the existence of S-integral solutions of simultaneous diophantine inequalities for pairs (Q, L) involving one quadratic form and one linear form satisfying some arithmetico-geometric conditions. This result generalises previous results of Gorodnik and Borel-Prasad. The proof uses Ratner's theorem for unipotent actions on homogeneous spaces combined with an argument of strong approximation.

Keywords: Quadratic forms, Diophantine approximation, Algebraic groups, Strong approximation, Ratner's Orbit Closure Theorem.

1. Introduction

A famous conjecture made by Oppenheim in 1929 and proved by G.A. Margulis in the mid-eighties states that given a nondegenerate indefinite real quadratic form Q in $n \geqslant 3$ variables which is not proportional to a form with rational coefficients then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}^n$. It is not difficult to see that the latter statement is equivalent to the following assertion,

$$\forall \varepsilon > 0, \exists\, x \in \mathbb{Z}^n - \{0\}, \quad 0 < |Q(x)| < \varepsilon\,. \tag{1}$$

A natural generalization of the Oppenheim conjecture concerns the existence of integral solutions of system of inequalities involving several quadratic forms. More precisely given a family $(Q_j)_{1 \leq j \leq r}$ of real nondegenerate quadratic forms in n variables we may ask whether there exist solutions to the diophantine inequalities

$$\forall \varepsilon > 0, \exists\, x \in \mathbb{Z}^n - \{0\},\ |Q_j(x)| < \varepsilon \ \text{ for } j = 1, \ldots, r. \tag{2}$$

Such kind of problems have been intensively studied and a general solution is still an open problem when $r > 1$. Some partial results have been obtained and almost all of them assume two fundamental necessary conditions for (2) to hold. The first condition is the existence of nonzero real solution $x \in \mathbb{R}^n$ ($n \geq 3$):

$$Q_1(x) = \ldots = Q_r(x) = 0.$$

The second condition is of arithmetical nature: it asks that for any $(\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r - \{0\}$, the pencil forms $\sum_{i=1}^r \alpha_i Q_i(x)$ are not proportional to a rational form. Notice that these two conditions are natural generalisation of the assumptions of the Oppenheim conjecture to the general case of systems of quadratic inequalities.