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New relations for the number of partitions with distinct even parts

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ABSTRACT

The number of partitions of n wherein even parts are distinct and odd parts are unrestricted, often denoted by $ped(n)$, has been the subject of many recent studies. In this paper, the author provides an efficient linear recurrence relation for $ped(n)$. A simple criterion for deciding whether $ped(n)$ is odd or even is given as a corollary of this result. Some connections with partitions into parts not congruent to 2 (mod 4), overpartitions and partitions into distinct parts are presented in this context.

1. Introduction

Let $ped(n)$ be the function which enumerates the number of partitions of n wherein even parts are distinct and odd parts are unrestricted. The generating function for $ped(n)$,

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty},$$

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was discussed in [1]. This is well known and appears in the following classic identity of Lebesgue

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n}{(q;q)_n}q^{\binom{n+1}{2}}=(-q^2;q^2)_\infty(-q;q)_\infty=\frac{(q^4;q^4)_\infty}{(q;q)_\infty},$$

where

$$(a;q)_n=\begin{cases}1, & \text{for } n=0,\\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{for } n>0\end{cases}$$

is the q-shifted factorial and

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a\neq 0$ and $|q|\geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q|<1$.

We remark that the sequence $\{ped(n)\}_{n\geqslant 0}$ is well known and can be seen in [17, A001935] together with other combinatorial interpretations. Recently, Andrews, Hirschhorn and Sellers [4], Chen [5], Cui and Gu [7], Hirschhorn and Sellers [11] and Xia [18] obtained many interesting congruences modulo 2, 3, 4, 6, 8 and 12 for $ped(n)$.

According to Fink, Guy and Krusemeyer [9], the numbers of partitions of n wherein even parts are distinct and odd parts are unrestricted satisfy Euler's recurrence relation for the partition function $p(n)$ unless n is four times a generalized pentagonal number, i.e.,

$$\sum_{j=-\infty}^{\infty}(-1)^k ped(n-j(3j+1)/2)=\begin{cases}(-1)^k, & \text{if } n=2k(3k+1),\ k\in\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}\tag{1}$$

In this paper, motivated by these results, we provide new recurrence relations for $ped(n)$ that involve the triangular numbers, i.e.,

$$T_k=k(k+1)/2,\quad k\in\mathbb{N}_0.$$

Theorem 1.1. *For $n\geqslant 0$,*

$$\sum_{j=0}^{\infty}(-1)^{\lceil j/2\rceil}ped(n-T_j)=\begin{cases}1, & \textit{if } n=2T_k,\ k\in\mathbb{N}_0,\\ 0, & \textit{otherwise.}\end{cases}$$

A more efficient recurrence relation for $ped(n)$ is given by the following result.