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Shortened recurrence relations for generalized Bernoulli numbers and polynomials

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ABSTRACT

It is the main purpose of this paper to study shortened recurrence relations for generalized Bernoulli numbers and polynomials attached to χ, χ being a primitive Dirichlet character, in which some of the preceding numbers or polynomials are completely excluded. As a result, we are able to establish several kinds of such type recurrences by generalizing some known identities on classical Bernoulli numbers and polynomials such as Saalschütz–Gelfand and von Ettingshausen–Stern's formulas. Furthermore, we discuss shortened recurrence relations for special values of the Riemann zeta and its allied functions.

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1. Introduction

Let B_n and $B_n(x)$, $n = 0, 1, 2, \ldots$, be the classical Bernoulli numbers and polynomials defined by the generating functions

$$\mathcal{F}(t) := \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi; \tag{1.1}$$

$$\mathcal{F}(t, x) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \tag{1.2}$$

respectively.

It is easy to see that $B_0 = 1$, $B_{2k+1} = 0$ and $(-1)^{k-1}B_{2k} > 0$ for all $k \geq 1$. Further, from the relation $\mathcal{F}(t, x) = \mathcal{F}(t)e^{xt}$, it is possible to express $B_n(x)$ as

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}.$$

Various types of linear recurrence relations for these numbers and polynomials have been developed over the years (cf. [15,12,9]). Among them, the most basic ones are

$$\sum_{i=0}^{n-1} \binom{n}{i} \frac{B_{i+1}}{i+1} + \frac{1}{n+1} = 0 \quad (n \geq 1); \tag{1.3}$$

$$\sum_{i=0}^{n-1} \binom{n}{i} \frac{B_{i+1}(x)}{i+1} + \frac{1}{n+1} = x^n \quad (n \geq 1). \tag{1.4}$$

These identities are easily obtained by expanding both sides of each of the functional relations $\mathcal{F}(t)(e^t - 1) = t$ and $\mathcal{F}(t, x)(e^t - 1) = te^{xt}$ into the Maclaurin power series and then comparing the coefficients of t^{n+1} on both sides, respectively.

Identity (1.3) involves all the consecutive Bernoulli numbers $B_1, B_2, \ldots, B_n$. In contrast to this, the following *shortened* (or *incomplete*) recurrence relation was discovered by Saalschütz [14], and later by M.B. Gelfand [8]:

$$\sum_{i=0}^{k} \binom{k}{i} \frac{B_{m+1+i}}{m+1+i} + (-1)^{k+m} \sum_{j=0}^{m} \binom{m}{j} \frac{B_{k+1+j}}{k+1+j} = \frac{(-1)^{m+1}}{k+m+1} \binom{k+m}{k}^{-1}, \tag{1.5}$$

which is valid for arbitrary integers $k, m \geq 0$. The most remarkable feature of (1.5) is that the first $\min\{k, m\}$ Bernoulli numbers are completely missing.

The study of such type recurrence relations has a long and interesting history, and it must go back to von Ettingshausen [21] in 1827 and Stern [18] in 1878. Indeed, they first discovered the following surprising and unusual formula that involves only the second half of Bernoulli numbers up to B_{2m+1}: