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Density measures and additive property

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ABSTRACT

We consider a certain class of normalized positive linear functionals on l^∞ which extend the Cesàro mean. We study the set of its extreme points and it turns out to be the set of linear functionals constructed from free ultrafilters on natural numbers $\mathbb{N}$. Also, regarding them as finitely additive measures defined on all subsets of $\mathbb{N}$, which are often called density measures, we study a certain additivity property of such measures being equivalent to the completeness of the L^p-spaces on such measures. Particularly a necessary and sufficient condition for such a density measure to have this property is obtained.

1. Introduction

Let l^∞ be the Banach space of all real-valued bounded functions on $\mathbb{N}$ and $L^\infty(\mathbb{R}_+^\times)$ be the Banach space of all real-valued essentially bounded measurable functions on $\mathbb{R}_+^\times = [1, \infty)$. Let $(l^\infty)^*$ and $L^\infty(\mathbb{R}_+^\times)^*$ be their conjugate spaces respectively. The symbol $\mathcal{P}(\mathbb{N})$ stands for the family of all subsets of $\mathbb{N}$, and for a set $A \in \mathcal{P}(\mathbb{N})$ let $|A|$ denote the cardinality of A. Recall that the Cesàro mean of a function $f \in l^\infty$ is defined as

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$$C(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i)$$

if this limit exists. When f is the characteristic function I_A of a set $A \in \mathcal{P}(\mathbb{N})$, its Cesàro mean $C(I_A) = d(A) = \lim_{n\to\infty} \frac{|A\cap[1,n]|}{n}$ is often called the asymptotic density of A. We consider a class $\mathcal{C}$ of normalized positive linear functionals on l^∞ concerning Cesàro summability method. Namely, linear functionals φ on l^∞ satisfying the following condition:

$$\varphi(f) \le \overline{C}(f) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i)$$

for each $f \in l^\infty$. It is remarked that such a functional φ is an extension of Cesàro mean, that is, $\varphi(f) = C(f)$ provided the limit exists. $\mathcal{C}$ is a compact convex set in its weak* topology and hence by the Krein–Milman theorem, the set of extreme points $ex(\mathcal{C})$ of $\mathcal{C}$ is not an empty set. An example of such a functional is given by

$$\varphi^{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim_{n} \frac{1}{n} \sum_{i=1}^{n} f(i),$$

where $f \in l^\infty$ and the limit in the above definition means the limit along a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (the precise definition of this notion is given in the following section). We denote the set of all such functionals by $\tilde{\mathcal{C}}$. Remark that there are distinct free ultrafilters $\mathcal{U}$ and $\mathcal{U}'$ which give the same element of $\tilde{\mathcal{C}}$, thus $\tilde{\mathcal{C}}$ is isomorphic as a set to some quotient space of the set of free ultrafilters on $\mathbb{N}$. We show that each functional in $\tilde{\mathcal{C}}$ is equal to some $\varphi^{\mathcal{U}}$ with $\mathcal{U}$ is a certain kind of ultrafilter, which has a form convenient to investigate the associated functional. The relation between $\tilde{\mathcal{C}}$ and $\mathcal{C}$ can be understood simply in view of the theory of linear topological spaces, that is, it is shown that $\tilde{\mathcal{C}}$ is precisely the set of extreme points $ex(\mathcal{C})$ of $\mathcal{C}$. Also we show that each element of $\mathcal{C}$ can be expressed as an integral with respect to some unique probability measure supported by its extreme points. It may be regarded as an interesting example concerning Choquet's theorem. For these purposes, it is useful to introduce an integral analogy $\mathcal{M}$ of $\mathcal{C}$ which is a class of normalized positive linear functionals on $L^\infty(\mathbb{R}_+^\times)$ defined by using the subadditive functional $\overline{M}$ on $L^\infty(\mathbb{R}_+^\times)$ which adopts the integral with respect to the Haar measure of real line $\mathbb{R}$ in place of the summation: namely, $\mathcal{M}$ is the set of linear functionals ψ on $L^\infty(\mathbb{R}_+^\times)$ for which

$$\psi(f) \le \overline{M}(f) = \limsup_{x\to\infty} \frac{1}{x} \int_1^x f(t)dt$$

holds for every $f \in L^\infty(\mathbb{R}_+^\times)$. Similarly we define a subclass $\tilde{\mathcal{M}}$ of $\mathcal{M}$ consisting of those $\psi^{\mathcal{U}}$ defined by