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Ternary quadratic form with prime variables attached to Fourier coefficients of primitive holomorphic cusp form

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A B S T R A C T

Let $\lambda_f(n)$ be the Fourier coefficients of primitive holomorphic cusp forms for $SL_2(\mathbb{Z})$. In this paper, we study the hybrid problem of ternary quadratic forms and Fourier coefficients of primitive holomorphic cusp form.

1. Introduction

Let f be a primitive holomorphic cusp form for the group $SL_2(\mathbb{Z})$ of even integral weight k, with Fourier coefficients $\lambda_f(n)$:

$$f(z) = \sum_{n>0} \lambda_f(n) n^{(k-1)/2} e(nz).$$

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It follows from the Ramanujan Conjecture that

$$|\lambda_f(n)| \le d(n),$$

where $d(n)$ is the Dirichlet divisor function (this result is due to Deligne [5]).

Many scholars are interested in the properties of the ternary quadratic form $m_1^2 + m_2^2 + m_3^2$. In 1963, Vinogradov [15] and Chen [4] independently studied the well-known sphere problem and showed the asymptotic formula

$$\sum_{\substack{m_1^2+m_2^2+m_3^2\le x\\ m_j\in\mathbb{Z}(j=1,2,3)}} 1 = \frac{4}{3}\pi x^{3/2} + O(x^{2/3}).$$

The exponent $2/3$ in the above error term was improved to $29/44$ by Chamizo and Iwaniec [2], and to $21/32$ by Heath-Brown [9].

Recently, several authors studied some problems connected with the ternary quadratic form. For example, let $\pi_3(x)$ denote the number of integer points

$$(m_1, m_2, m_3) \in \mathbb{Z}^3 \quad \text{with} \quad m_1^2 + m_2^2 + m_3^2 = p \le x.$$

Friedlander and Iwaniec [6] proved that

$$\pi_3(x) \sim \frac{4\pi}{3}\frac{x^{3/2}}{\log x},$$

which can be viewed as a generalization of the prime number theorem.

Guo and Zhai [7] studied the divisors of the quadratic form

$$S(x) := \sum_{1\le m_1,m_2,m_3\le x} d(m_1^2 + m_2^2 + m_3^2)$$

and obtained

$$S(x) = 2C_1I_1x^3 + (C_1I_2 + C_2I_1)x^3 + O(x^{8/3+\varepsilon}),$$

which improves the result of C. Calderon and M. J. de Velasco [1], where C_1, I_1, C_2, I_2 are computable constants. Later, Zhao [16] improved the error term to $O(x^2\log^3 x)$. Sun and Zhang [13] studied the triple divisors of ternary quadratic form

$$\sum_{1\le n_1,n_2,n_3\le\sqrt{x}} d_3(n_1^2 + n_2^2 + n_3^2) = c_1x^{\frac{3}{2}}(\log x)^2 + c_2x^{\frac{3}{2}}\log x + c_3x^{\frac{3}{2}} + O(x^{\frac{11}{8}}),$$

where $d_3(n)$ is the number of solutions of the equation $l_1l_2l_3 = n$ in natural numbers. Also Guo and Zhai [7] stated that