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Variation on a theme of Nathan Fine. New weighted partition identities

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Dedicated to our friend, Krishna Alladi, on his 60th birthday

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We utilize false theta function results of Nathan Fine to discover four new partition identities involving weights. These relations connect Göllnitz–Gordon type partitions and partitions with distinct odd parts, partitions into distinct parts and ordinary partitions, and partitions with distinct odd parts where the smallest positive integer that is not a part of the partition is odd and ordinary partitions subject to some initial conditions, respectively. Some of our weights involve new partition statistics, one is defined as the number of different odd parts of a partition larger than or equal to a given value and another one is defined as the number of different even parts larger than the first integer that is not a part of the partition.

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1. Introduction

A *partition*, $\pi = (\lambda_1, \lambda_2, \dots)$, is a finite sequence of non-increasing positive integers. Let $\nu(\pi)$ be the number of elements of π. These elements, λ_i for $i \in \{1, \dots, \nu(\pi)\}$, are called *parts* of the partition π. The *norm* of a partition π, denoted $|\pi|$, is defined as the sum of all its parts. We call a partition π *a partition of n* if $|\pi| = n$. Conventionally, we define the empty sequence to be the unique partition of zero. Also define $\nu_d(\pi)$ as the number of different parts of π. For example, $\pi = (10, 10, 5, 5, 4, 1)$ is a partition of 35 with $\nu(\pi) = 6$ and $\nu_d(\pi) = 4$.

While the study of the classical partition identities goes back to great Euler, the study of weighted partition identities is relatively new with many important consequences to be discovered. In 1997, Alladi [1] began a systematic study of weighted partition identities. Among many interesting results, he proved that

Theorem 1.1 *(Alladi, 1997).*

$$\frac{(a(1-b)q;q)_n}{(aq;q)_n} = \sum_{\pi \in \mathcal{U}_n} a^{\nu(\pi)} b^{\nu_d(\pi)} q^{|\pi|}, \tag{1.1}$$

where $\mathcal{U}_n$ is the set of partitions with the largest part $\leq n$.

In (1.1) and in the rest of the paper we use the standard q-Pochhammer symbol notations defined in [8,16]. Let L be a non-negative integer, then

$$(a;q)_L := \prod_{i=0}^{L-1} (1 - aq^i) \text{ and } (a;q)_\infty := \lim_{L \to \infty} (a;q)_L.$$

Theorem 1.1 provides a combinatorial interpretation for the left-hand side product of (1.1) as a weighted count of ordinary partitions with a restriction on the largest part. In [14], Corteel and Lovejoy elegantly interpreted (1.1) with $a = 1$ and $b = 2$,

$$\frac{(-q;q)_n}{(q;q)_n} = \sum_{\pi \in \mathcal{U}_n} 2^{\nu_d(\pi)} q^{|\pi|}, \tag{1.2}$$

in terms of *overpartitions*.

Also in [1], Alladi discovered and proved a weighted partition identity relating unrestricted partitions and the Rogers–Ramanujan partitions. Let $\mathcal{U}$ be the set of all partitions, and let $\mathcal{RR}$ be the set of partitions with difference between parts ≥ 2.

Theorem 1.2 *(Alladi, 1997).*

$$\sum_{\pi \in \mathcal{RR}} \omega(\pi) q^{|\pi|} = \sum_{\pi \in \mathcal{U}} q^{|\pi|}, \tag{1.3}$$