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On a remarkable identity in class numbers of cubic rings

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ABSTRACT

In 1997, Y. Ohno empirically stumbled on an astoundingly simple identity relating the number of cubic rings $h(\Delta)$ of a given discriminant Δ, over the integers, to the number of cubic rings $\hat{h}(\Delta)$ of discriminant -27Δ in which every element has trace divisible by 3:

$$\hat{h}(\Delta) = \begin{cases} 3h(\Delta) & \text{if } \Delta > 0 \\ h(\Delta) & \text{if } \Delta < 0, \end{cases} \tag{1}$$

where in each case, rings are weighted by the reciprocal of their number of automorphisms. This allows the functional equations governing the analytic continuation of the Shintani zeta functions (the Dirichlet series built from the functions h and $\hat{h}$) to be put in self-reflective form. In 1998, J. Nakagawa verified (1). We present a new proof of (1) that uses the main ingredients of Nakagawa's proof (binary cubic forms, recursions, and class field theory), as well as one of Bhargava's celebrated higher composition laws, while aiming to stay true to the stark elegance of the identity.

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1. Introduction

Great progress has been made in recent years [15] in analyzing statistics pertaining to cubic fields, ordered by discriminant. A basic analytic tool at one's disposal is the *Shintani zeta functions*, a pair of Dirichlet series that encode the number of cubic *rings* over $\mathbb{Z}$ of each nonzero discriminant:

$$\zeta^{+}(s) = \sum_{\substack{C/\mathbb{Z} \text{ cubic,} \\ \operatorname{Disc} C>0}} \frac{(\operatorname{Disc} C)^{-s}}{|\operatorname{Aut} C|}$$

$$\zeta^{-}(s) = \sum_{\substack{C/\mathbb{Z} \text{ cubic,} \\ \operatorname{Disc} C<0}} \frac{(-\operatorname{Disc} C)^{-s}}{|\operatorname{Aut} C|}.$$

The division by the number of automorphisms is a standard trick in this discipline which ensures, among other things, that the relative weights of a ring and its subrings (some of which may be isomorphic) are in the proper ratio. Because almost all cubic fields (and rings) have trivial automorphism group, this factor has no effect in most analytic applications.

The Shintani zeta functions were introduced in 1972 by Shintani, who proved that they have meromorphic continuations to the whole complex plane satisfying a reflection formula of the form (see [10], eq. (0.1))

$$\begin{bmatrix} \zeta^{+}(1-s) \\ \zeta^{-}(1-s) \end{bmatrix} = \begin{bmatrix} c_1(s) & c_2(s) \\ c_3(s) & c_4(s) \end{bmatrix} \begin{bmatrix} \hat{\zeta}^{+}(s) \\ \hat{\zeta}^{-}(s) \end{bmatrix} \tag{2}$$

connecting them to two other Dirichlet series $\hat{\zeta}^{+}$ and $\hat{\zeta}^{-}$ (the c_i, which are certain elementary expressions involving the Γ function, need not detain us). The functions $\hat{\zeta}^{+}$ and $\hat{\zeta}^{-}$ arise as follows. Call a cubic ring *integer-matrix*, or $\mathbb{Z}$*-mat* for short, if the trace of each of its elements is a multiple of 3. (This name will be demystified in the next section.) The discriminant of such a ring is always divisible by 27, making the scaling of the following Dirichlet series natural:

$$\hat{\zeta}^{+}(s) = 3^{3s} \sum_{\substack{C/\mathbb{Z}\ \mathbb{Z}\text{-mat,} \\ \operatorname{Disc} C>0}} \frac{(\operatorname{Disc} C)^{-s}}{|\operatorname{Aut} C|}$$

$$\hat{\zeta}^{-}(s) = 3^{3s} \sum_{\substack{C/\mathbb{Z}\ \mathbb{Z}\text{-mat,} \\ \operatorname{Disc} C<0}} \frac{(-\operatorname{Disc} C)^{-s}}{|\operatorname{Aut} C|}.$$

Shintani's functional equation stood unimproved until 1997, when Y. Ohno computed the first 200 terms of each of the four zeta functions and conjectured that they are equal in pairs, up to a curiously sign-dependent scale factor [12]: