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An infinite family of pairs of imaginary quadratic fields with both class numbers divisible by five

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We construct a new infinite family of pairs of imaginary quadratic fields with both class numbers divisible by five. Let n be a positive integer that satisfy $n \equiv \pm 3 \pmod{500}$ and $n \not\equiv 0 \pmod{3}$. We prove that 5 divides the class numbers of both $\mathbb{Q}(\sqrt{2 - F_n})$ and $\mathbb{Q}(\sqrt{5(2 - F_n)})$, where F_n is the nth Fibonacci number.

1. Introduction

Some infinite families of quadratic fields with class numbers divisible by a fixed integer N were given by Nagell [15], Ankeny and Chowla [1], Yamamoto [19], Weinberger [18], Gross and Rohrich [5], Ichimura [6] and Louboutin [13]. In the case $N = 5$, some results

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are known due to Parry [16], Mestre [14], Sase [17] and Byeon [3]. One of the authors [10], by using the Fibonacci numbers F_n, gave an infinite family of imaginary quadratic fields with class numbers divisible by five: the $\mathbb{Q}(\sqrt{-F_n})$ with $n \equiv 25 \pmod{50}$.

Recently, Komatsu [11,12] and Ito [9] (resp. Iizuka, Konomi and Nakano [7]) gave infinite families of pairs of quadratic fields with both class numbers divisible by 3 (resp. 3, 5 or 7). In the present article, by using the Fibonacci numbers F_n, we will give an infinite family of pairs of imaginary quadratic fields with both class numbers divisible by 5.

Theorem. *For $n \in \mathcal{N} := \{n \in \mathbb{N} \mid n \equiv \pm 3 \pmod{500}, n \not\equiv 0 \pmod{3}\}$, the class numbers of both $\mathbb{Q}(\sqrt{2-F_n})$ and $\mathbb{Q}(\sqrt{5(2-F_n)})$ are divisible by 5. Moreover, the set of pairs $\{(\mathbb{Q}(\sqrt{2-F_n}), \mathbb{Q}(\sqrt{5(2-F_n)})) \mid n \in \mathcal{N}\}$ is infinite.*

For an algebraic extension K/k, denote the norm map and the trace map of K/k by $N_{K/k}$ and $\mathrm{Tr}_{K/k}$, respectively. For simplicity, we denote N_K and Tr_K if the base field is $k = \mathbb{Q}$. For a prime number p and an integer m, we denote the greatest exponent μ of p such that $p^\mu \mid m$ by $v_p(m)$.

2. Certain parametric quartic polynomial

Let $k = \mathbb{Q}(\sqrt{5})$. For an algebraic integer $\alpha \in k$, we consider the polynomial

$$f(X) = f_\alpha(X) := X^4 - TX^3 + (N+2)X^2 - TX + 1 \in \mathbb{Z}[X], \tag{2.1}$$

where $T := \mathrm{Tr}_k(\alpha)$ and $N := N_k(\alpha)$. The discriminant of $f(X)$ is $\mathrm{disc}(f) = d_1^2 d_2$ with $d_1 := T^2 - 4N$ and $d_2 := (N+4)^2 - 4T^2$. Let L be the minimal splitting field of $f(X)$ over $\mathbb{Q}$. All four complex roots of $f(X)$ are units of L and can be denoted by $\varepsilon, \varepsilon^{-1}, \eta, \eta^{-1}$, $|\varepsilon| \geq |\varepsilon^{-1}|$, $|\eta| \geq |\eta^{-1}|$, $\alpha = \varepsilon + \varepsilon^{-1}$, $\overline{\alpha} = \eta + \eta^{-1}$, where $\overline{\alpha}$ denotes the Galois conjugate of α (see [2, Lemmas 2.2 and 2.3]). We assume $\alpha \notin \mathbb{Z}$, $\alpha^2 - 4 \notin \mathbb{Z}^2$, $d_2 \in 5\mathbb{Q}^2$ and $\alpha^2 - 4 > 0$. The assumptions $\alpha \notin \mathbb{Z}$ and $\alpha^2 - 4 \notin \mathbb{Z}^2$ imply that the polynomial $f(X)$ is $\mathbb{Q}$-irreducible, and we have $\mathrm{Gal}(L/\mathbb{Q}) \simeq C_4$ from $d_2 \in 5\mathbb{Q}^2$ (see [2, Proposition 2.1]). Furthermore, we have $\varepsilon, \eta \in \mathbb{R}$ by the assumption $\alpha^2 - 4 > 0$, $d_2 > 0$ and the factorization

$$f(X) = (X^2 - \alpha X + 1)(X^2 - \overline{\alpha} X + 1) = (X - \varepsilon)(X - \varepsilon^{-1})(X - \eta)(X - \eta^{-1}) \tag{2.2}$$

(see [2, Lemma 2.7]). Set $\widetilde{L} = L(\zeta_5)$ where ζ_5 is a primitive fifth root of unity. Since $\mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \supset \mathrm{Gal}(\widetilde{L}/k) \simeq C_2 \times C_2$ and $\mathrm{Gal}(\widetilde{L}/\mathbb{Q})/\mathrm{Gal}(\widetilde{L}/\mathbb{Q}(\zeta_5)) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \simeq C_4$, we have $\mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \simeq C_2 \times C_4$. Therefore, $\mathrm{Gal}(\widetilde{L}/\mathbb{Q})$ has three subgroups of order 4. One of them is isomorphic to $C_2 \times C_2$ that corresponds to the subfield k, the others are isomorphic to C_4. Let us denote them by $\langle \tau \rangle$ ($\simeq C_4$) and $\langle \tau' \rangle$ ($\simeq C_4$) for some automorphisms $\tau, \tau' \in \mathrm{Gal}(\widetilde{L}/\mathbb{Q})$ of order 4. Note that $\zeta_5^\tau \neq \zeta_5, \zeta_5^4$, because τ acts trivial