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On zero-sum subsequences of length not exceeding a given number

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ABSTRACT

Let G be an additive finite abelian group. For a positive integer k, let $\mathsf{s}_{\leq k}(G)$ denote the smallest integer l such that each sequence of length l has a non-empty zero-sum subsequence of length at most k. Among other results, we determine $\mathsf{s}_{\leq k}(G)$ for all finite abelian groups of rank two.

1. Introduction

Let C_n denote the cyclic group of n elements. Let G be an additive finite abelian group. It is well known that $|G| = 1$ or $G = C_{n_1} \oplus C_{n_2} \cdots \oplus C_{n_r}$ with $1 < n_1|n_2 \cdots |n_r$. Then, $r(G) = r$ is the rank of G and the exponent $\exp(G)$ of G is n_r. Let

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$$S := g_1 \cdots g_l$$

be a sequence with elements in G. We call S a zero-sum sequence if $g_1 + \cdots + g_l = 0$. The Davenport constant $\mathsf{D}(G)$ is the minimal integer $l \in N$ such that every sequence S over G of length $|S| \geq l$ has a nonempty zero-sum subsequence. Set

$$\mathsf{D}^*(G) := 1 + \sum_{i=1}^{r}(n_i - 1).$$

Then $\mathsf{D}(G) \geq \mathsf{D}^*(G)$. Let $\eta(G)$ denote the smallest integer $l \in N$ such that every sequence S over G of length $|S| \geq l$ has a nonempty zero-sum subsequence T of length $|T| \leq \exp(G)$. In this paper, we investigate a following generalization of $\mathsf{D}(G)$ and $\eta(G)$.

Definition 1. Denote by $\mathsf{s}_{\leq k}(G)$ the smallest element $l \in N \cup \{+\infty\}$ such that each sequence of length l has a non-empty zero-sum subsequence of length at most k ($k \in N$).

The constant $\mathsf{s}_{\leq k}(G)$ was introduced by Delorme, Ordaz and Quiroz [2]. It is a special case for a more general definition of zero-sum constant given by Geroldinger, Grynkiewicz and Schmid [5]. It is trivial to see that $\mathsf{s}_{\leq k}(G) = \mathsf{D}(G)$ if $k \geq \mathsf{D}(G)$, $\mathsf{s}_{\leq k}(G) = \eta(G)$ if $k = \exp(G)$ and $\mathsf{s}_{\leq k}(G) = \infty$ if $1 \leq k < \exp(G)$. In general, the problem of determining $\mathsf{s}_{\leq k}(G)$ is not at all trivial. Recently, the exact number of $\mathsf{s}_{\leq 3}(C_2^r)$ is known by the work of Freeze and Schmid [3], namely, $1+2^{r-1}$. Besides its own interesting, Cohen and Zemor [1] pointed out a connection between $\mathsf{s}_{\leq k}(C_2^r)$ and coding theory. In this paper, we shall determine $\mathsf{s}_{\leq k}(G)$ for some groups. Our main results are the following:

Theorem 2. *Let $G = C_m \oplus C_n$, where m and n are integers with $1 \leq m|n$. Then*

$$\mathsf{s}_{\leq \mathsf{D}(G)-k}(G) = \mathsf{D}(G) + k \textit{ for all } k \in [0, m-1].$$

Theorem 3. *Let $G = C_2^r$ for some $r \in \mathbb{N}$. Then*

$$\mathsf{s}_{\leq r-k}(G) = r + 2 \textit{ for all } r - k \in \left[\left\lceil \frac{2r+2}{3} \right\rceil, r\right].$$

2. Preliminaries

In this paper, our notations are coincident with [4,6] and we briefly present some key concepts. Let N denote the set of positive integers and $N_0 = N \cup \{0\}$.

Let $\mathscr{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathscr{F}(G)$ are called sequences over G. Let

$$S = g_1 \cdots g_l \in \mathscr{F}(G),$$