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A Hermite–Minkowski type theorem of varieties over finite fields

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ABSTRACT

We show the finiteness of étale coverings of a variety over a finite field with given degree whose ramification bounded along an effective Cartier divisor. The proof is an application of P. Delgine's theorem (H. Esnault and M. Kerz, 2012) [4] on a finiteness of l-adic sheaves with restricted ramification. By applying our result to a smooth curve over a finite field, we obtain a function field analogue of the classical Hermite–Minkowski theorem.

1. Introduction

For a number field F, that is, a finite extension of $\mathbb{Q}$, the Hermite–Minkowski theorem asserts that there exist only finitely many extensions of the number field F with given degree unramified outside a finite set of primes of F (e.g., [15], Chap. III, Thm. 2.13; [5], Chap. V, Thm. 2.6). In [5], G. Faltings gave a higher dimensional generalization of this theorem stated as follows:

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Theorem 1.1 *([5], Chap. VI, Sect. 2.4; [9], Thm. 2.9). Let X be a connected scheme of finite type and dominant (e.g., flat) over* $\mathrm{Spec}(\mathbb{Z})$*. Then there exist only finitely many étale coverings of X with given degree.*

Here, an **étale covering** of X means a finite étale morphism $X' \to X$. The aim of this note is to give a "function field" analogue of this theorem. We begin simple observations:

- For a function field F of one variable over a finite field with characteristic p, the Artin–Schreier equations produce infinitely many extensions of F of degree p which ramify only in a finite set of places (see e.g., [7], Sect. 8.23).
- For a number field F, (the exponents of) the discriminant of an extension of F has an upper bound depending on the extension degree and the primes at which it ramifies ([16], Chap. III, Sect. 6, Prop. 13, see also remarks after the proposition). Under the conditions in the Hermite–Minkowski theorem, namely, the extension degree and a finite set of primes are given, the discriminants of extensions of F are automatically bounded.

Considering these facts together, to obtain a finiteness as above in the case of function fields we have to restrict ramification.

Now, we present the results in this note more precisely. Let X be a connected and separated scheme of finite type over a finite field (we call such schemes just **varieties** in the following *cf.* Notation), and $\overline{X}$ a compactification of X (*cf.* Sect. 2). For an effective Cartier divisor D with support $|D| \subset Z = \overline{X} \smallsetminus X$, we will introduce the notion of *bounded ramification along* D for étale coverings of X (whose ramification locus is in the boundary Z) in the next section (Definition 2.2). Adopting this notion, we show the following theorem.

Theorem 1.2 *(Theorem 3.4). Let $X \subset \overline{X}$ be as above. There exist only finitely many étale coverings of X with bounded degree and ramification bounded by a given effective Cartier divisor D with support in $Z = \overline{X} \smallsetminus X$.*

A key ingredient for the proof is (a weak form of) Deligne's finiteness theorem on smooth Weil sheaves with bounded ramification [4] (Theorem 3.3).

For an étale covering $X' \to X$ of smooth *curves* over a finite field, if its degree and the discriminant are bounded, then the ramification of the covering $X' \to X$ in our sense is also bounded (Proposition 2.9):

$$\text{bounded degree \& discriminant} \;\Rightarrow\; \text{bounded ramification.}$$

From this, we obtain an alternative proof of the following well-known theorem: