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Ce Xu, Zhonghua Li

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Tornheim type series and nonlinear Euler sums

Ce Xu[*] Zhonghua Li[†]

∗ School of Mathematical Sciences, Xiamen University
Xiamen 361005, P.R. China
† School of Mathematical Sciences, Tongji University
Shanghai 200092, P.R. China

Abstract In this paper, we develop an approach to evaluation of nonlinear Euler sums. The approach is based on Tornheim type series computations. By the approach, we can obtain some closed form representations of quadratic and cubic sums in terms of zeta values and linear sums. Furthermore, we also evaluate several other series involving harmonic numbers. Some interesting new consequences and illustrative examples are considered.

Keywords Tornheim type series; harmonic numbers; polylogarithm function; Euler sums; Riemann zeta function.

AMS Subject Classifications (2010): 11M06; 11M32; 33B99; 33E20; 40A05

1 Introduction

In this paper, the n-th generalized harmonic number of order k, denoted by $H_n^{(k)}$, is defined by

$$H_n^{(k)} := \sum_{j=1}^{n} \frac{1}{j^k},\ n,k \in \mathbb{N} := \{1,2,3,\ldots\}, \tag{1.1}$$

where $H_n^{(1)} = H_n := \sum\limits_{j=1}^{n} \frac{1}{j}$ is the n-th harmonic number. For any $k \in \mathbb{N}$, we set $H_0^{(k)} = 0$. The generalized harmonic number converges to the Riemann zeta value $\zeta(k)$:

$$\lim_{n\to\infty} H_n^{(k)} = \zeta(k),\ \Re(k) > 1,\ k \in \mathbb{N},$$

where the Riemann zeta function is defined by (for more details, see for instance, [2, 5, 6])

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1.$$

The classical linear Euler sum is defined by

$$S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} = \sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{k=1}^{n} \frac{1}{k^p}, \tag{1.2}$$

[*]Corresponding author. Email: 15959259051@163.com (C. Xu)
[†]zhonghua li@tongji.edu.cn (Z.H. Li)