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Chaotic and topological properties of continued fractions

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We prove that there exists a scrambled set for the Gauss map with full Hausdorff dimension, in particular, it is chaotic in the sense of Li–Yorke. Meanwhile, we also investigate the topological properties of the sets of points with dense or non-dense orbits.

1. Introduction

It is known that every irrational number $x \in [0, 1)$ can admit an infinite continued fraction (CF) induced by Gauss transformation $T : [0, 1) \to [0, 1)$ given by

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ for } x \in (0,1) \text{ and } T(0) := 0$$

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where $\lfloor y \rfloor$ denotes the integer part of a real number y, that is, $\lfloor y \rfloor = n$, if $y \in [n, n+1)$ for some $n \in \mathbb{Z}$. We set

$$a_1(x) := \lfloor x^{-1} \rfloor \quad \text{and} \quad a_n(x) := \lfloor (T^{n-1}(x))^{-1} \rfloor, \ \ n \geq 2,$$

and have the following CF expansion of x:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}} = [a_1(x), a_2(x), a_3(x), \cdots].$$

The numbers $a_n(x)$ $(n \geq 1)$ are called the partial quotients of x. In 1845, Gauss observed that T preserves the probability measure given by

$$\mu(B) = \frac{1}{\log 2} \int_B \frac{1}{1+x} d\mathcal{L}(x),$$

where $B \subset [0, 1)$ is any Borel measurable set and $\mathcal{L}$ is the Lebesgue measure on $[0, 1]$. The measure μ is called Gauss measure and equivalent with $\mathcal{L}$.

Continued fractions are a kind of representation of real numbers and an important tool to study the Diophantine approximation in number theory. Many metric and dimensional results on Diophantine approximation have been obtained with the help of continued fractions such as Good [5], Jarnik [7], Khintchine [9] etc., and also the dimensional properties of continued fractions were considered, for example Wang and Wu [21], Xu [26] etc. The behaviors of the continued fraction dynamical systems have been widely investigated, for example, the shrinking target problems [14], mixing property [18], thermodynamic formalism [15], limit theorems [3] etc. Hu and Yu [6] were concerned with the set $\{x \in [0, 1) : \xi \notin \overline{\{T^n(x) : n \geq 0\}}\}$ for any $\xi \in [0, 1)$ and proved that it is 1/2-winning and thus it has full Hausdorff dimension.

Asymptotic behaviors of the orbits are one of the main theme in dynamical systems. In this paper, we focus on the continued fraction dynamical system $([0, 1), T)$ and study firstly the denseness of the orbits, and secondly the size of the scrambled set from the sense of Hausdorff dimension. Let

$$D := \{x \in [0, 1) : \text{the orbit of } x \text{ under } T \text{ is dense in } [0, 1]\ \},$$

and D^c be the complement of the set D in $[0, 1]$.

Theorem 1.1. *Let T be the Gauss map on $[0, 1)$. Then*
(1) *D is of full Lebesgue measure in $[0, 1]$.*
(2) *D^c is of full Hausdorff dimension.*