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THE DISTRIBUTION OF INTEGERS IN A TOTALLY REAL CUBIC FIELD

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Abstract. Hecke studies the distribution of fractional parts of quadratic irrationals with Fourier expansion of Dirichlet series. This method is generalized by Behnke and Ash-Friedberg, to study the distribution of the number of totally positive integers of given trace in a general totally real number field of any degree. When the field is cubic, we show that the asymptotic behavior of a weighted Diophantine sum is related to the structure of the unit group. The main term can be expressed in terms of Grössencharacter L-functions.

1. Introduction

The study of the equidistribution of the fractional part of $m\alpha$ for α irrational and $m = 1, 2, \ldots$ running over the rational integers, dates back to Weyl's work [9] in 1910. Hecke [8] studied the case when α is a fixed real quadratic irrational. His key idea is using the Fourier expansion of the Dirichlet series

$$\sum_{m\geq 1}\left(\{m\alpha\} - \frac{1}{2}\right)m^{-s}$$

to estimate the Diophantine sum

$$S_1(n) = \sum_{m=1}^{n}\left(\{m\alpha\} - \frac{1}{2}\right). \tag{1.1}$$

Both Behnke [4] and Ash-Friedberg [1] aim at generalizing Hecke's result to an arbitrary totally real field K of degree n. In such cases, the generalization of the fractional part of $m\alpha$ is the error term in the natural geometric estimate for the number of totally positive integers of K of a given trace. They form the Dirichlet series $\varphi(s)$ whose coefficients are these errors. More specifically, let O_K be the ring of integers of K, and let $\mathrm{Tr}(O_K)$ be generated by $\kappa > 0$. For positive multiples a of κ, let N_a denote the number of totally positive integers with trace a. There is the natural geometric estimate r_a of N_a derived from the volume of the intersection in $O_K \otimes \mathbb{R}$ of the cone of totally positive elements with the hyperplane defined by $\mathrm{Tr}\alpha = a$. Denote the difference between the true value and the estimate by

$$E_a = N_a - r_a. \tag{1.2}$$

If a is not a multiple of κ, we set $E_a = 0$. Then we define the Dirichlet series

$$\varphi(s) = \sum_{a>0}\frac{E_a}{a^s}.$$

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