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HIGH RANK QUADRATIC TWISTS OF PAIRS OF ELLIPTIC CURVES

MOHAMED ALAA AND MOHAMMAD SADEK

Abstract. Given a pair of elliptic curves E_1 and E_2 over the rational field $\mathbb{Q}$ whose j-invariants are not simultaneously 0 or 1728, Kuwata and Wang proved the existence of infinitely many square-free rationals d such that the d-quadratic twists of E_1 and E_2 are both of positive rank. We construct infinite families of pairs of elliptic curves E_1 and E_2 over $\mathbb{Q}$ such that for each pair there exist infinitely many square-free rationals d for which the d-quadratic twists of E_1 and E_2 are both of rank at least 2.

1. Introduction

Goldfeld Conjecture states that the average rank of elliptic curves over $\mathbb{Q}$ in families of quadratic twists is 1/2. This reflects the strong belief amongst number theorists that quadratic twists with rank at least 2 of an elliptic curve defined over $\mathbb{Q}$ are seldom. In fact, one may find a great deal of literature investigating the rank frequencies for quadratic twists of elliptic curves.

In [8, 9], Mestre proved that given an elliptic curve over $\mathbb{Q}$, there are infinitely many quadratic twists whose rank is at least 2. Furthermore, he introduced infinitely many elliptic curves with infinitely many quadratic twists whose rank is at least 3. He proved, moreover, that if E is an elliptic curve over $\mathbb{Q}$ whose torsion subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, then it has infinitely many quadratic twists with rank at least 4.

In [11, 12], quadratic twists of ranks at least 2 and 3 were introduced. In addition, infinitely many elliptic curves with infinitely many quadratic twists of rank at least 4 were constructed. For the latter families, the quadratic twists are parametrized by an elliptic curve of positive rank.

Many of the families of elliptic curves that were constructed in [11, 12] are families of Legendre elliptic curves. Those are elliptic curves described by the equation $y^2 = x(x-1)(x-\lambda)$ where $\lambda \in \mathbb{Q} - \{0, 1\}$. In [4], more families of Legendre elliptic curves with infinitely many quadratic twists of rank at least 2 were displayed.

In [7], the study of quadratic twists of pairs of elliptic curves over $\mathbb{Q}$ was initiated. Given a pair of elliptic curves E_1 and E_2 over the rational field $\mathbb{Q}$ whose j-invariants are not simultaneously 0 or 1728, it was proved that there exist infinitely many square-free rational numbers d such that the d-quadratic twists of E_1 and E_2 are both of positive rank. Similar questions were posed in [2]. Examples of infinitely many pairs of elliptic curves E_1 and