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Pairs of quadratic forms over a quadratic field extension

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ABSTRACT

Let F be a field of characteristic distinct from 2, $L = F(\sqrt{d})$ a quadratic field extension. Let further f and g be quadratic forms over L considered as polynomials in n variables, M_f, M_g their matrices. We say that the pair (f, g) is a k-pair if there exist $S \in GL_n(L)$ such that all the entries of the $k \times k$ upper-left corner of the matrices SM_fS^t and SM_gS^t are in F. We give certain criteria to determine whether a given pair (f, g) is a k-pair. We consider the transfer $\mathrm{cor}_{L(t)/F(t)}$ determined by the $F(t)$-linear map $s : L(t) \to F(t)$ with $s(1) = 0$, $s(\sqrt{d}) = 1$, and prove that if $\dim \mathrm{cor}_{L(t)/F(t)}(f + tg)_{an} \leq 2(n - k)$, then (f, g) is a $[\frac{k+1}{2}]$-pair. If, additionally, the form $f + tg$ does not have a totally isotropic subspace of dimension $p + 1$ over $L(t)$, we show that (f, g) is a $(k - 2p)$-pair. In particular, if the form $f + tg$ is anisotropic, and $\dim \mathrm{cor}_{L(t)/F(t)}(f + tg)_{an} \leq 2(n - k)$, then (f, g) is a k-pair.

Let F be a field, $\mathrm{char}\, F \neq 2$, $d \in F^* \setminus F^{*2}$, $L = F(\sqrt{d})$, V an n-dimensional linear space over L. A quadratic form $V \to L$ over L will be called an L-form. Let further $s : L \to F$ be the F-linear map determined by $s(1) = 0$, $s(\sqrt{d}) = 1$. For an L-form $\varphi : V \to L$ denote by $\mathrm{cor}_{L/F}(\varphi)$ (corestriction) the F-form $V \xrightarrow{\varphi} L \xrightarrow{s} F$, where V is considered as an F-vector space. This $2n$-dimensional form is usually called the transfer of φ determined by the map s ([6], Ch. 2, §5). If the field extension L/F is clear from the context, we omit the symbol L/F and simply write $\mathrm{cor}(\varphi)$.

Recall some properties of this transfer. First, it is easy to check, following the definition of the transfer that

$$\mathrm{cor}_{L/F}(\langle a + b\sqrt{d}\rangle) \simeq \langle b, -bN_{L/F}(a + b\sqrt{d})\rangle,$$

if $a, b \in F$, $b \neq 0$. Moreover, $\mathrm{cor}(\langle a\rangle) \simeq \mathbb{H}$ if $a \in F^*$, and $\mathrm{cor}(\langle 0\rangle) \simeq \langle 0, 0\rangle$. These equalities permit to compute $\mathrm{cor}(\varphi)$ if one knows a diagonal presentation of φ. In particular, if φ is regular, then so is $\mathrm{cor}(\varphi)$. Furthermore, obviously, the transfer respects the direct sum of the forms, i.e. $\mathrm{cor}(\varphi_1 \perp \varphi_2) \simeq \mathrm{cor}(\varphi_1) \perp \mathrm{cor}(\varphi_2)$ for any L-forms φ_1, φ_2. Also there is the projection formula. Namely, let U be a finite-dimensional linear space

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over F. For an F-form $\varphi : U \to F$ consider the form $\varphi_L : U \otimes_F L \to L$ defined in the obvious way. It is easy to verify that $\operatorname{cor}(\varphi_L \otimes \psi) \simeq \varphi \otimes \operatorname{cor}\psi$ for any F-form φ and L-form ψ ([6], Ch. 2, 5.6). Finally, by Elman–Lam's theorem ([3]) the anisotropic part $\operatorname{cor}(\varphi)_{an}$ of the form $\operatorname{cor}(\varphi)$ is zero if and only if φ is defined over F, i.e. all the coefficients of φ with respect to some basis of V are in F. The following is an easy generalization of the last statement.

Proposition 1. *For any $0 \le k \le n$ the following two conditions are equivalent:*

1) *There is a subform $\varphi_0 \subset \varphi$ such that φ_0 is defined over F and $\dim\varphi_0 = k$.*
2) $\dim \operatorname{cor}(\varphi)_{an} \le 2(n-k)$.

Proof. Recall first that any form ψ over an arbitrary field of characteristic not 2 has a decomposition $\psi \simeq s\langle 0\rangle \perp p\mathbb{H} \perp \psi_{an}$ with uniquely determined s, p, and (up to isometry) anisotropic part ψ_{an}. Moreover, $s+p$ is exactly the dimension of any maximal totally isotropic subspace of ψ.

1) $\Longrightarrow$ 2) Let $\varphi_0 \simeq r\langle 0\rangle \perp \varphi_1$, where φ_1 is regular, defined over F, and $r \le k$. Then $2r\langle 0\rangle \perp (k-r)\mathbb{H} \simeq \operatorname{cor}(\varphi_0) \subset \operatorname{cor}(\varphi)$. In particular, $(k-r)\mathbb{H} \subset \operatorname{cor}(\varphi)$. Since the form $(k-r)\mathbb{H}$ is regular, we get that $(k-r)\mathbb{H}$ is a direct summand of $\operatorname{cor}(\varphi)$. Hence $\operatorname{cor}(\varphi) \simeq s\langle 0\rangle \perp p\mathbb{H} \perp \operatorname{cor}(\varphi)_{an}$ for some p, s, where $p \ge k-r$. Since $\operatorname{cor}(\varphi_0) \subset \operatorname{cor}(\varphi)$, and $\operatorname{cor}(\varphi_0)$ has a totally isotropic subspace of dimension $2r + (k-r) = k+r$, we get that $p+s \ge k+r$. Therefore, $s+2p = p+(p+s) \ge (k-r)+(k+r) = 2k$. Thus,

$$\dim \operatorname{cor}(\varphi)_{an} = \dim \operatorname{cor}(\varphi) - s - 2p = 2n - s - 2p \le 2(n-k).$$

2) $\Longrightarrow$ 1) Assume first that φ is anisotropic and then induct on k. If $k=0$, the claim is obvious. Assume that $k \ge 1$, i.e. the form $\operatorname{cor}(\varphi) : V \xrightarrow{\varphi} L \xrightarrow{s} F$ is isotropic. By definition of the transfer this means that there exists a vector $0 \ne v \in V$ such that $\varphi(v) \in F^*$. Therefore, $\varphi \simeq \langle \varphi(v)\rangle \perp \widetilde{\varphi}$ for some form $\widetilde{\varphi}$. Since $\dim(\operatorname{cor}\widetilde{\varphi})_{an} = \dim(\operatorname{cor}\varphi)_{an} \le 2(n-1-(k-1))$, we get by the induction hypothesis a form $\widetilde{\varphi}_0 \subset \widetilde{\varphi}$, where $\widetilde{\varphi}_0$ is defined over F, and $\dim\widetilde{\varphi}_0 = k-1$. Hence we can put $\varphi_0 = \langle\varphi(v)\rangle \perp \widetilde{\varphi}_0$.

In the general case let $\varphi \simeq r\langle 0\rangle \perp m\mathbb{H} \perp \psi$, where ψ is anisotropic. Then $\dim\psi = n-2m-r$, and

$$\dim(\operatorname{cor}\psi)_{an} = \dim(\operatorname{cor}\varphi)_{an} \le 2((n-2m-r)-(k-2m-r)).$$

If $k-2m-r \ge 0$, then in view of the anisotropic case we get $\psi_0 \subset \psi$, where the form ψ_0 is defined over F, and $\dim\psi_0 = k-2m-r$. Thus, we can put $\varphi_0 \simeq r\langle 0\rangle \perp m\mathbb{H} \perp \psi_0$. If $k-2m-r < 0$, then we can take any k-dimensional F-subform of $r\langle 0\rangle \perp m\mathbb{H}$ as φ_0. $\square$

Choosing a basis of V we can consider the form φ as a homogeneous quadratic polynomial $\varphi(x_1,\dots,x_n)$ with coefficients from L. In this situation we will often identify the form φ with its symmetric matrix of the coefficients $M_\varphi = (a_{ij})$, where $\varphi = \sum_{i,j} a_{ij}x_ix_j$ with $a_{ij} = a_{ji}$. The equivalent conditions in Proposition 1 mean that there exists a linear change of variables $(x_1,\dots,x_n) = (x'_1,\dots,x'_n)S$, where $S \in GL_n(L)$, such that the coefficients at the monomials $x'_ix'_j$, $(1 \le i,j \le k)$ for the form $\varphi((x'_1,\dots,x'_n)S)$ belong to F. In other words, all the entries of the $k\times k$ upper-left corner of the matrix $SM_\varphi S^t$ are in F.

The main purpose of this paper is to get a generalization of Proposition 1 for a *pair* of forms. Namely, consider the following question. Let f and g be forms in n variables over L, M_f, M_g their matrices, and $1 \le k \le n$. When does there exist $S \in GL_n(L)$ such that all the entries of the $k\times k$ upper-left corners of the matrices SM_fS^t and SM_gS^t are in F? If S exists, the pair of forms (f,g) will be called k-pair, and if $k=n$, the pair (f,g) is said to be defined over F. Also we say that (f,g) is a k-pair if $k \le 0$. Clearly, if (f,g) is a k-pair, then the form $f+tg$ has a subform of dimension k defined over $F(t)$, so $\dim \operatorname{cor}_{L(t)/F(t)}(f+tg)_{an} \le 2(n-k)$ by Proposition 1.

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