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Finitistic dimension conjecture and radical-power extensions

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ABSTRACT

The finitistic dimension conjecture asserts that any finite-dimensional algebra over a field has finite finitistic dimension. Recently, this conjecture is reduced to studying finitistic dimensions for extensions of algebras. In this paper, we investigate those extensions of Artin algebras in which some radical-power of smaller algebras is a nonzero one-sided ideal of bigger algebras. Our result can be formulated for an arbitrary ideal as follows: Let $B \subseteq A$ be an extension of Artin algebras and I an ideal of B such that the full subcategory of B/I-modules is B-syzygy-finite. (1) If the extension is right-bounded (for example, $\mathrm{Gpd}(A_B) < \infty$), $I\,A\,\mathrm{rad}(B) \subseteq B$ and $\mathrm{findim}\,(A) < \infty$, then $\mathrm{findim}\,(B) < \infty$. (2) If $I\,\mathrm{rad}(B)$ is a left ideal of A and A is torsionless-finite, then $\mathrm{findim}\,(B) < \infty$. Particularly, if I is specified to a power of the radical of B, then our result not only generalizes some of results in the literature (see Corollary 1.2), but also provides new ways to detect algebras of finite finitistic dimensions.

1. Introduction

Let A be an Artin algebra. The finitistic dimension of A is defined to be the supremum of the projective dimensions of finitely generated left A-modules having finite projective dimension. The famous finitistic dimension conjecture says that any Artin algebra has finite finitistic dimension (see [3, Conjecture (11), p. 410]). This conjecture was initially a question by Rosenberg and Zelinsky, published by Bass in a paper in 1960 (see [4]), and has attracted many mathematicians in the last 5.5 decades. Among them is Maurice Auslander who "is considered to be one of the founders of the modern aspects of the representation theory of

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Artin algebras" (see [14, p. 501]). "One of his main interests in the theory of Artin algebras was the finitistic dimension conjecture and related homological conjectures" (see [14, p. 815]). The conjecture has intimate connections with the solutions of several other open conjectures such as Nakayama conjecture, generalized Nakayama conjecture and Gorenstein symmetry conjecture (see [21] for a survey on these conjectures). Though several special cases for the conjecture to be true have been verified (see, for example, [8,9,11,12, 22,23,5]), it is still open in general. Actually, up to the present time, not many practical methods, so far as we know, are available to detect algebras of finite finitistic dimension. It seems necessary to develop some methods for testing finiteness of finitistic dimensions for general algebras or even for some concrete examples.

In the paper [17, Corollary 4.8], the conjecture has been reduced to comparing finitistic dimensions of a pair of algebras instead of focusing only on one single algebra. More precisely, the following two statements are proved to be equivalent for a field k:

(1) The finitistic dimension of any finite-dimensional k-algebra is finite.

(2) For any extension $B \subseteq A$ of finite-dimensional k-algebras such that $\mathrm{rad}(B)$, the Jacobson radical of B, is a left ideal in A, if A has finite finitistic dimension, then B has finite finitistic dimension.

Along this line, the conjecture is further reduced, by a different method, to extensions of algebras with relative global dimension 1, where the ground field is assumed to be perfect (see [20, Proposition 2.9]). Thus it seems quite worthy to consider such kinds of extensions of algebras and to bound the finitistic dimensions of smaller algebras in terms of the ones of bigger algebras which we would like to take as simple as possible.

Let us just mention a few considerations in this direction. In the sequel, we denote by $\mathrm{gldim}\,(A)$ and $\mathrm{findim}\,(A)$ the global and finitistic dimensions of an algebra A, respectively; and by $\mathrm{pd}\,(A_B)$ the projective dimension of the right B-module A.

(i) Let $B \subseteq A$ be an extension of Artin algebras such that $\mathrm{rad}(B)$ is a left ideal in A. Then:
   (a) If $\mathrm{rad}(A) = \mathrm{rad}(B)A$ and $\mathrm{gldim}\,(A) \leq 4$, then $\mathrm{findim}\,(B) < \infty$ (see [18, Theorem 3.7]).
   (b) If $\mathrm{pd}\,(A_B) < \infty$ and $\mathrm{findim}\,(A) < \infty$, then $\mathrm{findim}\,(B) < \infty$ (see [20, Corollary 1.4]).

(ii) Let $C \subseteq B \subseteq A$ be a chain of Artin algebras such that $\mathrm{rad}(C)$ and $\mathrm{rad}(B)$ are left ideals in B and A, respectively. If A is representation-finite, then $\mathrm{findim}\,(C) < \infty$ (see [17, Theorem 4.5]).

Thus the philosophy of controlling finitistic dimensions by extension algebras works powerfully (see also the example at the end of Section 3).

In this paper, we continue to explore and compare finitistic dimensions of extensions $B \subseteq A$ of Artin algebras. Since it happens quite often that $\mathrm{rad}(B)$ itself may not be a left ideal in A but some of its powers are actually nonzero left or right ideals in A, our main goal in this paper is to extend results for the case that $\mathrm{rad}(B)$ is a left ideal in A to a more general case that $\mathrm{rad}^s(B)$ is a left (or right) ideal in A for some positive integer s. Such kinds of extensions may be called *radical-power* extensions.

We shall carry out our discussion for extensions $B \subseteq A$ in a broader context by studying an arbitrary ideal of B rather than a power of the radical of B. The technical problem we encounter, even in the case of a higher power of the radical, is that the higher syzygies of B-modules admit no longer A-module structures. So a crucial ingredient for bounding projective dimensions used in [17,18,20] is missing. To circumvent this problem here, we first use certain submodules of torsionless B-modules to get A-module structures, and then establish certain short exact sequences connecting B-syzygies with the lifted A-modules. Finally, we employ the Igusa–Todorov function in [12] to estimate upper bounds of projective dimensions.

To state our main result more precisely, let us first introduce some definitions.

Let A be an Artin algebra and M_A be a right A-module. For an A-module X, we define $t_M(X) := inf\{n \in \mathbb{N} \mid \mathrm{Tor}_j^A(M, X) = 0 \text{ for all } j \geq n+1\}$, and $g_M(A) := sup\{t_M(X) \mid X \in A\text{-mod with } \mathrm{pd}\,({}_AX) < \infty\}$. We call $g_M(A)$ the *Gorenstein index* of A relative to M. In case of an extension $B \subseteq A$ of Artin algebras, we call $g_{A_B}(B)$ the Gorenstein index of the extension.

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