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Presenting Hecke endomorphism algebras by Hasse quivers with relations

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ABSTRACT

A Hecke endomorphism algebra is a natural generalisation of the q-Schur algebra associated with the symmetric group to a Coxeter group. For Weyl groups, B. Parshall, L. Scott and the first author [9,10] investigated the stratification structure of these algebras in order to seek applications to representations of finite groups of Lie type. In this paper we investigate the presentation problem for Hecke endomorphism algebras associated with arbitrary Coxeter groups. Our approach is to present such algebras by quivers with relations. If R is the localisation of $\mathbb{Z}[q]$ at the polynomials with the constant term 1, the algebra can simply be defined by the so-called idempotent, sandwich and extended braid relations. As applications of this result, we first obtain a presentation of the 0-Hecke endomorphism algebra over $\mathbb{Z}$ and then develop an algorithm for presenting the Hecke endomorphism algebras over $\mathbb{Z}[q]$ by finding torsion relations. As examples, we determine the torsion relations required for all rank 2 groups and the symmetric group $\mathfrak{S}_4$.

1. Introduction

Since I. Schur first introduced Schur algebras for linking representations of the general linear group $GL_n(\mathbb{C})$, a continuous group, with those of the symmetric group $\mathfrak{S}_r$, a finite group, this class of algebras has been generalised to various objects in several directions. For example, by the role they play in the Schur–Weyl duality, there are affine and super generalisations and their quantum analogues, which are known as (affine, super) q-Schur algebras (see, e.g., [20,5,15,24]). As homomorphic images of a universal enveloping algebra, generalised Schur algebras and their quantum analogues are investigated not only for

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type A but also for other types (see, e.g., [6,11]). By their definition as endomorphism algebras of certain permutation modules of symmetric groups, they have a natural generalisation to Hecke endomorphism algebras associated with finite Coxeter groups (see, e.g., [8–10,12]).

Recently, with a completely different motivation, advances have been made in categorifying Hecke endomorphism algebras. For example, M. Mackaay et al. obtained a diagrammatic categorification of the q-Schur algebra [22] and some affine counterpart [23], while G. Williamson [28] investigated a more general class of Hecke categories associated with any Coxeter systems (W, S) (called Schur algebroids [28]), and categorified them in terms of singular Soergel bimodules. As seen from these works, a presentation of an algebra by generators and relations is related to the categorification of Hecke-endomorphism algebras. In fact, the question of presenting the Hecke endomorphism algebras by generators and relations was raised in Remark 2.2 in [28].

Unlike the q-Schur algebras, Hecke endomorphism algebras do not have a direct connection to Lie algebras and quantum groups and thus cannot be presented as a quotient of a quantum group. However, by viewing them as enlarged Hecke algebras, they can be presented with generators and relations, which are rooted in Hecke (algebra) relations. This paper is going to tackle the presentation problem in this direction.

Our approach is to present these algebras by quivers with relations, following closely the idea of using Hecke relations. First, by assuming the invertibility of all Poincaré polynomials in the ground ring R, we consider in Sections 3 and 4 a quiver Q' with loops and impose the Hecke relations directly on the loops. We then replace the loops by cycles of length 2 to obtain a quiver Q with no loops and braid relations are imposed explicitly again. Now, by thinking of the generators displayed in [28, Cor. 2.12], we take a subset labelled by $I, J \subseteq S$ satisfying $I \sqsubset J$ (see (2.5.1)) and introduce in Section 5 a Hasse quiver $\tilde{Q}$ to replace Q. It turns out that the algebra over R can simply be presented by $\tilde{Q}$ together with the so-called extended braid relations and some obvious relations, called idempotent relations and sandwich relations (see Theorem 5.1).

The first application of this result is to obtain a presentation of the 0-Hecke endomorphism algebras by specialising q to 0 in Section 6. These degenerate algebras have attracted some attention (see, e.g., [25,2,13,26,18,19,17,4]) and have also nice applications. For instance, J. Stembridge [27] used the 0-Hecke algebra to give a derivation of the Möbius function of the Bruhat order, while X. He [16] gave a more elementary construction of a monoid structure by A. Berenstein and D. Kazhdan [1].

We then investigate the integral case in Sections 7 and 8. We analyse the gap between the presentation over R and a possible presentation over the polynomial ring $\mathbb{Z}[q]$. The idempotent relations can be replaced by the so-called quasi-idempotent relations, the sandwich relations are unchanged. The challenge is how to replace the extended braid relations by some torsion relations. We develop an algorithm and compute the examples of rank 2 and of type A_3. The rank 2 case is relatively easy, the required torsion relations are simply the refined braid relations. Note that a recursive version of this case is done by B. Elias [14, Prop. 2.20]. However, the A_3 case is more complicated. On top of the refined braid relations, there are two more sets of torsion relations, see Theorem 8.3. We believe that the algorithm can be used to compute the other lower rank cases.

2. Hecke algebras and Kazhdan–Lusztig generators

Let (W, S) be a Coxeter system and let $\ell : W \to \mathbb{N}$ be the length function with respect to S and $\leq$ the Bruhat order.

For $I \subseteq S$, let $W_I = \langle s \mid s \in I\rangle$ be the parabolic subgroup generated by I. We say $I \subseteq S$ is *finitary* if W_I is finite, and in this case we denote by w_I the longest element in W_I. Let

$$\Lambda = \Lambda(W) = \{I \subseteq S \mid I \text{ is finitary}\} \quad \text{and} \quad \Lambda^* = \Lambda \setminus \emptyset,$$

where $\emptyset \in \Lambda$ is the empty set.