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Efficient generation of ideals in a discrete Hodge algebra

Manoj K. Keshari, Md. Ali Zinna *

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

A R T I C L E I N F O

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A B S T R A C T

Let R be a commutative Noetherian ring and D be a discrete Hodge algebra over R of dimension $d > \dim(R)$. Then we show that
(i) the top Euler class group $E^d(D)$ of D is trivial,
(ii) if $d > \dim(R) + 1$, then $(d-1)$-st Euler class group $E^{d-1}(D)$ of D is trivial.

1. Introduction

Let R be a commutative Noetherian ring. An R-algebra D is called a *discrete Hodge algebra over R* if $D = R[X_1, \cdots, X_n]/\mathcal{I}$, where $\mathcal{I}$ is an ideal of $R[X_1, \cdots, X_n]$ generated by monomials. Typical examples are $R[X_1, \cdots, X_n]$, $R[X, Y](XY)$, etc. In [19], Vorst studied the behavior of projective modules over discrete Hodge algebras. He proved [19, Theorem 3.2] that *every finitely generated projective D-module is extended from R if for all k, every finitely generated projective $R[X_1, \cdots, X_k]$-module is extended from R.*

Later Mandal [15] and Wiemers [20] studied projective modules over discrete Hodge algebra D. In [20], Wiemers proved the following significant result. *Let P be a projective D-module of rank $\geq \dim(R) + 1$. Then (i) $P \simeq Q \oplus D$ for some D-module Q and (ii) P is cancellative, i.e. $P \oplus D \simeq P' \oplus D$ implies $P \simeq P'$.*

When $D = R[X, Y](XY)$, the above results of Wiemers are due to Bhatwadekar and Roy [2]. Very recently, inspired by results of Bhatwadekar and Roy, Das and Zinna [11] studied the behavior of ideals in $R[X, Y]/(XY)$ and proved the following result on efficient generation of ideals. *Assume $\dim(R) \geq 1$, $D = R[X, Y]/(XY)$ and $I \subset D$ is an ideal of height $n = \dim(D)$. Assume I/I^2 is generated by n elements. Then any given set of n generators of I/I^2 can be lifted to a set of n generators of I. In particular, the top Euler class group $E^n(D)$ of D is trivial.*

As $R[X, Y]/(XY)$ is the simplest example of a discrete Hodge algebra over R, motivated by above discussions, one can ask the following question.

* Corresponding author.
E-mail addresses: keshari@math.iitb.ac.in (M.K. Keshari), zinna@math.iitb.ac.in (Md.A. Zinna).

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Question 1.1. Let R be a commutative Noetherian ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n > \dim(R)$. Let $I \subset D$ be an ideal of height n. Suppose that $I = (f_1, \cdots, f_n) + I^2$. Do there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$? In other words, is the top Euler class group $E^n(D)$ of D trivial? (For definition of Euler class groups, see [4] and [5].)

We answer Question 1.1 affirmatively and prove the following more general result ((3.1) below).

Proposition 1.2. *Let R be a commutative Noetherian ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $n > \dim(R)$. Let P be a projective D-module of rank n which is extended from R and I be an ideal in D of height ≥ 2. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$. In particular, the n-th Euler class group $E^n(D)$ of D is trivial.*

The above result can be extended to any rank n projective D-module when R contains $\mathbb{Q}$ (3.8). Here is the precise statement.

Theorem 1.3. *Let R be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension ≥ 2 and D be a discrete Hodge algebra over R of dimension $n > \dim(R)$. Let I be an ideal in D of height ≥ 3 and P be any rank n projective D-module. Suppose that there is a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$.*

After studying the top rank case, one is tempted to go one step further and inquire the following question.

Question 1.4. Let R be a commutative Noetherian ring of dimension ≥ 3 and D be a discrete Hodge algebra over R of dimension $d \geq \dim(R) + 2$. Let I be an ideal in D of height $d - 1$ and P be a projective D-module of rank $d - 1$. Suppose that $\alpha : P/IP \twoheadrightarrow I/I^2$ is a surjection. Can α be lifted to a surjection $\beta : P \twoheadrightarrow I$?

We answer Question 1.4 affirmatively when R contains $\mathbb{Q}$ ((4.3) below) as follows.

Theorem 1.5. *Let R be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension ≥ 3 and D be a discrete Hodge algebra over R of dimension $d > \dim(R)$. Let I be an ideal in D of height ≥ 4 and P be a projective D-module of rank $n \geq \max\{\dim(R) + 1, d - 1\}$. Suppose that $\alpha : P/IP \twoheadrightarrow I/I^2$ is a surjection. Then there exists a surjection $\beta : P \twoheadrightarrow I$ which lifts α. As a consequence, if $d \geq \dim(R) + 2$, then $(d - 1)$-st Euler class group $E^{d-1}(D)$ of D is trivial.*

Finally we derive an interesting consequence of above result as follows (see (4.6)).

Theorem 1.6. *Let R be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension ≥ 3 and D be a discrete Hodge algebra over R of dimension $d > \dim(R)$. Let I be a locally complete intersection ideal in D of height $n \geq \max\{\dim(R) + 1, d - 1\}$. Then I is set theoretically generated by n elements.*

In Section 5, we give some partial answer to the following question.

Question 1.7. Let R be a commutative Noetherian ring of dimension ≥ 1 and D be a discrete Hodge algebra over R of dimension $> \dim(R)$. Let $I \subset D$ be an ideal of height $> \dim(R)$. Suppose that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq \dim(D/I) + 2$. Do there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$?

The above question has been settled in the affirmative by Mandal in [14] when D is a polynomial algebra over R. Recently Fasel [12] has settled a conjecture of Murthy and proved the following result. *Let k be an infinite field of characteristic $\neq 2$ and $I \subset k[T_1, \cdots, T_m]$ be an ideal. Then we have $\mu(I) = \mu(I/I^2)$.*

Therefore, we may ask the following natural question.

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