Accepted Manuscript

On the complexity of the permanent in various computational models

Christian Ikenmeyer, J.M. Landsberg

PII: S0022-4049(17)30033-6
DOI: http://dx.doi.org/10.1016/j.jpaa.2017.02.008
Reference: JPAA 5626

To appear in: *Journal of Pure and Applied Algebra*

Received date: 2 October 2016
Revised date: 1 February 2017

Please cite this article in press as: C. Ikenmeyer, J.M. Landsberg, On the complexity of the permanent in various computational models, *J. Pure Appl. Algebra* (2017), http://dx.doi.org/10.1016/j.jpaa.2017.02.008

On the complexity of the permanent in various computational models

Christian Ikenmeyer and J.M. Landsberg*

February 20, 2017

Abstract

We answer a question in [14], showing the regular determinantal complexity of the determinant $\det_m$ is $O(m^3)$. We answer questions in, and generalize results of [2], showing there is no rank one determinantal expression for perm_m or $\det_m$ when $m \geq 3$. Finally we state and prove several "folklore" results relating different models of computation.

1 Introduction

Let $P(y^1, \ldots, y^M) \in S^m\mathbb{C}^M$ be a homogeneous polynomial of degree m in M variables. A *size n determinantal expression* for P is an expression:

$$P = \det_n(\Lambda + \sum_{j=1}^{M} X^j y^j). \tag{1}$$

where X^j, Λ are $n \times n$ complex matrices.

The *determinantal complexity* of P, denoted $\text{dc}(P)$, is the smallest n for which a size n determinantal expression exists for P. Valiant [22] proved that for any polynomial P, $\text{dc}(P)$ is finite. Let $(y^{i,j})$, $1 \leq i, j \leq m$, be linear coordinates on the space of $m \times m$ matrices. Let $\text{perm}_m := \sum_{\sigma \in \mathfrak{S}_m} y^{1,\sigma(1)} \cdots y^{m,\sigma(m)}$ where $\mathfrak{S}_m$ is the permutation group on m letters.

Valiant's famous algebraic analog of the $\mathbf{P} \neq \mathbf{NP}$ conjecture [22] is:

Conjecture 1.1 (Valiant [22])**.** *The sequence* $\text{dc}(\text{perm}_m)$ *grows super-polynomially fast.*

The state of the art regarding determinantal expressions for perm_m is $2^m - 1 \geq \text{dc}(\text{perm}_m) \geq \frac{m^2}{2}$, respectively [8, 17].

In the same paper [22], Valiant also made the potentially stronger conjecture that there is no polynomial sized arithmetic circuit computing perm_m.

There are two approaches towards conjectures such as Conjecture 1.1. One is to first prove them in *restricted models*, i.e., assuming extra hypotheses, with the goal of proving a conjecture by first proving it under weaker and weaker supplementary hypotheses until one arrives at the original conjecture. The second is to fix a complexity measure such as $\text{dc}(\text{perm}_m)$ and then to prove lower bounds on the complexity measure, which we will call *benchmarks*, and then

*Landsberg partially supported by NSF grant DMS-1405348.