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Rigidity of down-up algebras with respect to finite group coactions

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ABSTRACT

If G is a nontrivial finite group coacting on a graded noetherian down-up algebra A inner faithfully and homogeneously, then the fixed subring $A^{co\ G}$ is not isomorphic to A. Therefore graded noetherian down-up algebras are rigid with respect to finite group coactions, in the sense of Alev–Polo. An example is given to show that this rigidity under group coactions does not have all the same consequences as the rigidity under group actions.

0. Introduction

Throughout this paper, let $\Bbbk$ be a base field that is algebraically closed of characteristic zero, and let all vector spaces, (co)algebras, and morphisms be over $\Bbbk$.

A remarkable theorem of Alev–Polo [1, Theorem 1] states:

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two semisimple Lie algebras. Let G be a finite group of algebra automorphisms of the universal enveloping algebra $U(\mathfrak{g})$ such that the fixed subring $U(\mathfrak{g})^G$ is isomorphic to $U(\mathfrak{g}')$. Then G is trivial and $\mathfrak{g} \cong \mathfrak{g}'$.

Alev–Polo called this result a rigidity theorem for universal enveloping algebras. In addition, they proved a rigidity theorem for the Weyl algebras [1, Theorem 2]. Kuzmanovich and the second- and third-named authors proved Alev–Polo's rigidity theorems in the graded case in [12, Theorem 0.2 and Corollary 0.4].

(Commutative) polynomial rings are not rigid; indeed, by the classical Shephard–Todd–Chevalley Theorem if G is a reflection group acting on a commutative polynomial ring A then A^G is isomorphic to A. Artin–Schelter regular algebras [2] are considered to be a natural analogue of polynomial rings in many respects. This paper concerns a class of noncommutative Artin–Schelter regular algebras. The rigidity of a noncommutative algebra is closely related to the lack of reflections in the noncommutative setting [12].

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Therefore the rigidity of an algebra leads to a trivialization of the Shephard–Todd–Chevalley theorem [19, 13], which is one of the key results in noncommutative invariant theory [10]. The rigidity property is also related to Watanabe's criterion for the Gorenstein property, see [14, Theorem 4.10]. Some recent work in noncommutative algebraic geometry connects the rigidity property and the lack of reflections to Auslander's theorem [4], which is one of the fundamental ingredients in the McKay correspondence [8,9]. Further understanding of the rigidity property will have implications for several other research directions.

Noncommutative algebras generally possess a rather limited set of groups actions, so exploring their (quantum) symmetries is accomplished by expanding the class of actions, and considering the actions of finite dimensional Hopf algebras. An obvious class of Hopf algebra actions to consider is the actions by the duals of group algebras, $(\Bbbk G)^*$, (or equivalently, group coactions of G). Note that the Hopf algebra $(\Bbbk G)^*$ is also denoted by $\Bbbk^G$ and called the functions on G by other authors. In [16], the study of rigidity with respect to group coactions was begun. Let A be a connected ($\mathbb{N}$-)graded $\Bbbk$-algebra. A G-coaction on A (preserving the $\mathbb{N}$-grading) is equivalent to a G-grading of A (compatible with the original $\mathbb{N}$-grading), and the fixed subring A^{coG} is A_e, the component of the unit element $e \in A$ under the G-grading.

Recall from [3] that a Hopf algebra H action on an $\mathbb{N}$-graded algebra A is *inner faithful* if there is no non-zero Hopf ideal I of H with $IA = 0$. When H is a semisimple Hopf algebra, by [18, Theorem 7] this condition is equivalent to the free algebra $\Bbbk\langle A\rangle$ being a faithful H-module; when A is generated in degree 1, this condition is equivalent to every simple H-module appearing in the semisimple decomposition of the free algebra generated by A_1

$$\Bbbk \oplus A_1 \oplus (A_1 \otimes A_1) \oplus (A_1 \otimes A_1 \otimes A_1) \oplus \cdots$$

as an H-module. Since the simple modules in $(\Bbbk G)^*$ correspond to group elements in G, a $(\Bbbk G)^*$-action on A is inner faithful if and only if the G-degrees of the generating set of A as an algebra generate G as a group. Throughout we assume that all $(\Bbbk G)^*$-actions (and hence G-coactions) are inner faithful.

We recall a definition [16, Definition 0.8]: we say that a connected graded algebra A is *rigid with respect to group coactions* if for every nontrivial finite group G coacting on A homogeneously and inner faithfully, the fixed subring $A^{co\,G}$ is not isomorphic to A as algebras. The following Artin–Schelter regular algebras are rigid with respect to group coactions [16, Theorem 0.9]:

(a) The homogenization of the universal enveloping algebra of a finite dimensional semisimple Lie algebra.
(b) The Rees ring of the Weyl algebra with respect to the standard filtration.
(c) The non-PI Sklyanin algebras of global dimension at least 3.

These results can be viewed as dual versions of the rigidity theorems proved in [12, Theorem 0.2 and Corollary 0.4].

Down-up algebras were introduced by Benkart–Roby in [5] as a tool to study the structure of certain posets. Graded noetherian down-up algebras are Artin–Schelter regular algebras of global dimension three with two generators [17]. We recall the definition of only a graded noetherian down-up algebra. For α and β scalars in $\Bbbk$, let the *down-up algebra* $\mathbb{D}(\alpha, \beta)$ be the algebra generated by u and d and subject to the relations

$$u^2 d = \alpha u d u + \beta d u^2, \tag{E0.0.1}$$

$$u d^2 = \alpha d u d + \beta d^2 u. \tag{E0.0.2}$$

When $\alpha = 0$ we denote the down-up algebra $\mathbb{D}(0, \beta)$ by $\mathbb{D}_\beta$. In this paper we always assume that $\beta \neq 0$, or equivalently, $\mathbb{D}(\alpha, \beta)$ is a graded noetherian Artin–Schelter regular algebra of global dimension three. The groups of algebra automorphisms of down-up algebras (which depend upon the values of α and β)