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The Divisibility Graph of finite groups of Lie type

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ABSTRACT

The *Divisibility Graph* of a finite group G has vertex set the set of conjugacy class sizes of non-central elements in G and two vertices are connected by an edge if one divides the other. We determine the connected components of the Divisibility Graph of the finite groups of Lie type in odd characteristic.

1. Introduction

Given a set X of positive integers, several graphs corresponding to X can be defined. For example the *prime vertex graph* $\Gamma(X)$, the *common divisor graph* $\Delta(X)$ and the *bipartite divisor graph* $B(X)$ (see [24,26] and references therein for more details). In 2011 Camina and Camina [8] introduced the *divisibility graph* $\mathrm{D}(X)$ of X as the directed graph with vertex set $X \setminus \{1\}$ with an edge from vertex a to vertex b whenever a divides b. For a group G let $cs(G)$ denote the set of conjugacy class sizes of non-central elements in G. Camina and Camina asked [8, Question 7] how many components the divisibility graph of $cs(G)$ has. Clearly it is sufficient to consider the underlying undirected graph and for the remainder of this paper the *Divisibility Graph* $\mathrm{D}(G)$ of a group G refers to the undirected Divisibility Graph $\mathrm{D}(cs(G))$.

Note that the set of vertices $cs(G)$ may be replaced by the set $\mathcal{C}(G)$ of orders of the centralisers of non-central elements of G.

In [7,26] has been shown that the graphs $\Gamma(cs(G)), \Delta(cs(G))$ and $B(cs(G))$ have at most two connected components when G is a finite group. Indeed, when G is a nonabelian finite simple group, then $\Gamma(cs(G))$ is complete (see [5,17]). It is clear that $\mathrm{D}(G)$ is a subgraph of $\Gamma(cs(G))$ and we hope that the structure of $\mathrm{D}(G)$ reveals more about the group G.

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The first and second authors have shown that for every comparability graph, there is a finite set X such that this graph is isomorphic to $\mathrm{D}(X)$ in [3]. They found some relationships between the combinatorial properties of $D(X), \Gamma(X)$ and $\Delta(X)$ such as the number of connected components, diameter and girth (see [3, Lemma 1]) and found a relationship between $\mathrm{D}(X \times Y)$ and product of $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ in [3]. They examined the Divisibility Graph $\mathrm{D}(G)$ of a finite group G in [1] and showed that when G is the symmetric or alternating group, then $\mathrm{D}(G)$ has at most two or three connected components, respectively. In both cases, at most one connected component is not a single vertex.

Here we are interested in the Divisibility Graphs of finite groups of Lie type. The graph of certain finite simple groups of Lie type is known, namely the first and second authors described in [2] the structures of the Divisibility Graphs for $\mathrm{PSL}(2, q)$ and $Sz(q)$. Let K_i denote the complete graph on i vertices. They prove in [2, Theorem 6] that for $G = \mathrm{PSL}(2, q)$ the graph $\mathrm{D}(G)$ is either $3\mathrm{K}_1$ or $\mathrm{K}_2 + 2\mathrm{K}_1$. For the other finite groups of Lie type in odd characteristic we prove the following theorem:

Theorem 1. *Let G be a finite group of Lie type over a finite field of order q or q^2 in characteristic p where p is an odd prime. Suppose further that the Lie rank ℓ of G is either as in Table 1 or at least 2. Then the Divisibility Graph $\mathrm{D}(G)$ has at most one connected component which is not a single vertex.*

In particular, we prove that the non-trivial connected component of $\mathrm{D}(G)$ contains the sizes of the conjugacy classes of all non-central involutions as well as the sizes of all conjugacy classes of all unipotent elements. The number of connected components consisting of a single vertex corresponds to the number of conjugacy classes T^G of tori T in G for which $|TZ(G)/Z(G)|$ is odd, coprime to $|Z(G)|$ and for which the centraliser in G of every non-central element in T is $TZ(G)$.

We now compare the results of the theorem to known results about another type of graph, namely the Prime Graph first introduced by Gruenberg and Kegel in 1975 in an unpublished manuscript. The vertex set of the *Prime Graph* of a finite group G is the set of primes dividing the order of the group and two vertices r and s are adjacent if and only if G contains an element of order rs. Williams [36, Lemma 6] investigated Prime Graphs of finite simple groups in odd characteristic and Kondrat'ev [25] and Lucido [28] investigated these graphs for even characteristic and for almost simple groups, respectively. A subgroup T of a group G is called a CC-group if $C_G(t) \leq T$ for all $t \in T \backslash Z(G)$. Williams proved [36, Theorem 1] that the connected components of the Prime Graph of a finite simple group G of Lie type in odd characteristic p consist at most of the set $\{p\}$, the connected component containing the prime 2, and a collection of sets consisting of the primes dividing the order of some torus T for which $TZ(G)/Z(G)$ has odd order coprime to $|Z(G)|$ and is a CC-group (see Section 3.3). Moreover, for such groups he showed that $\{p\}$ is an isolated vertex of the Prime Graph if and only if $G \cong \mathrm{PSL}(2, q)$ with q odd. Hence for the groups we consider (see Section 2.1 where we exclude small dimensions and certain 'bad' primes) we may assume that $\{p\}$ is not an isolated vertex in the Prime Graph. In this case, Williams shows that the number of connected components of a finite simple group of Lie type is at most two, unless $G = {}^2D_p(3)$ and $p = 2^n + 1$ a prime for $n \geq 2$, for which the Prime Graph has three components and unless $G = E_8(q)$, for which the Prime Graph can have either four or five components, depending on whether $q \equiv 2, 3 \pmod 5$ or $q \equiv 0, 1, 4 \pmod 5$, respectively. We verified the main theorem separately for the case of small dimensions or the 'bad' primes. Hence we obtain the following corollary.

Corollary 2. *Let G be a finite group of Lie type over a finite field of order q or q^2 in characteristic p where p is an odd prime. Suppose further that the Lie rank ℓ of G is either as in Table 1 or at least 2 and that $q \neq 3$ if G is of type E_7. Then the Divisibility Graph $\mathrm{D}(G)$ has as many connected components as the Prime Graph of G.*

We note that if G is of adjoint or simply connected type $E_7(3)$, the Divisibility Graph of G is connected, whereas the Prime Graph has three connected components (see [36, Table Id]).

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