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The transvection free groups with polynomial rings of invariants

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A R T I C L E I N F O

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A B S T R A C T

Let G be a finite linear group containing no transvections. This paper proves that the ring of invariants of G is polynomial if and only if the pointwise stabilizer in G of any subspace is generated by pseudoreflections. Kemper and Malle used the classification of finite irreducible groups generated by pseudoreflections to prove the irreducible case in arbitrary characteristic. We extend their result to the reducible case.

1. Introduction

The question of determining linear groups with polynomial rings of invariants has a long history. It was shown by Shephard–Todd [10] and Chevalley [1] that a finite linear group in characteristic zero has a polynomial ring of invariants if and only if it is generated by pseudoreflections. However, there are modular pseudoreflection groups with nonpolynomial rings of invariants. Serre [9] showed that in arbitrary characteristic if a finite linear group has a polynomial ring of invariants, then the pointwise stabilizer of any subspace also has a polynomial ring of invariants (see Theorem 2.10). This important result gives one direction of the assertion of Theorem 1.1 in this paper. Kac [4, Remark 3] conjectured a sufficient condition for a representation to have a polynomial ring of invariants. However, the transvection group $Sp_4(2)$ and the reflection group $RU_3(3)$ both show that it dose not suffice to require that the pointwise stabilizer of any subspace is generated by pseudoreflections in order to obtain a polynomial ring of invariants and thus refute Kac's conjecture. The paper of Nakajima [8] started the systematic investigation of irreducible groups generated by pseudoreflections. But he left out the case of characteristic 2, small ranks and the exceptional groups. As in the work of Nakajima, Kemper and Malle [5] used the classification of finite irreducible groups generated by pseudoreflections due to Kantor [3], Wagner [13–15], Zalesskii and Serezkin [16,17] to obtain a complete list of all irreducible linear groups with polynomial rings of invariants. Moreover, their proof also yielded the characterization of irreducible transvection free groups with polynomial rings of invariants. It may be worth

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noting that Kemper and Malle had used Serre's theorem incorrectly in [5]. In fact, the convention adopted in [5] is that the polynomial ring is $S(V)$ rather than $S(V^*)$. They had to reconsider all cases where they had applied Serre's theorem. Luckily, it turned out that the final results all remain correct. (see [2, Section 3.7.4])

In this paper, by the classification of finite irreducible pseudoreflection groups, we characterize transvection free groups and extend Kemper and Malle's result (see Theorem 2.7) for the irreducible transvection free groups with polynomial rings of invariants to the reducible case:

Theorem 1.1. *Let V be a finite dimensional vector space, $G \leq GL(V)$ a finite linear group containing no transvections and $S(V^*)$ symmetric algebra of V^*. Then $S(V^*)^G$ is polynomial if and only if the pointwise stabilizer in G of any subspace of V is generated by pseudoreflections.*

Here G acts on V^* by $g(l) = l \circ g^{-1}$ for $g \in G$, $l \in V^*$. This action is homomorphically extended to $S(V^*)$. As observed by Kemper and Malle, the criterion of the Theorem 1.1 is equivalent to the following: the stabilizer of any point in $\bar{V} := V \otimes_k K$ is generated by pseudoreflections, where K denotes an algebraic closure of the ground field k.

Section 2 of this paper gives the classification of finite irreducible linear groups generated by pseudoreflections and Kemper and Malle's results for the irreducible case. In Section 3, we prove Theorem 1.1.

It would be very interesting to find a priori proof of Theorem 1.1.

2. The finite irreducible linear groups with polynomial rings of invariants

In this section we recall the work of Kemper and Malle [5]. The material is partly copied from [5, Section 1 and Section 2]. Let k be a field of characteristic p and V a finite dimensional vector space over k. An element $\sigma \in GL(V)$ is called a *pseudoreflection* if $ker(1-\sigma)$ is a hyperplane in V. A pseudoreflection is called a *transvection* if it is not diagonalizable. A diagonalizable pseudoreflection is called a *homology*. A homology of order 2 is also called a *reflection*. We consider a finite irreducible linear group $G \leq GL(V)$ generated by pseudoreflections, and denote by k_G the minimal field of definition of G.

Suppose $G \leq GL(V)$ is an imprimitive irreducible group generated by pseudoreflections, then in some basis the group G has monomial form (see [16, 1.8]). Thus G is obtainable as a reduction modulo p from a imprimitive group in characteristic 0. We denote it by $G_p(m, l, n)$, $l \mid m$, in accordance with the notation of Shephard and Todd [10].

We denote the subgroup of $GL_n(q)$ (respectively $GU_n(q)$) generated by its reflections by $RL_n(q)$ (resp. $RU_n(q)$). ${\Omega_n}^{(\pm)}(q)$ denotes the commutator subgroup of the orthogonal group. $\mathfrak{S}_n$ denotes the symmetric group. $W_p(G_i)$ denotes the p-modular reduction of the primitive complex pseudoreflection group with number i in the list of Shephard and Todd [10]. In positive characteristic, the irreducible subgroups of $GL_n(k)$, $n \geq 3$, generated by reflections were classified by Wagner [15], Zalesskii and Serezkin [17].

Theorem 2.1 *(Wagner, Zalesskii and Serezkin). Let V be a vector space of dimension $n \geq 3$ over a finite field of characteristic $p \neq 2$ and $G \leq GL(V)$ an irreducible primitive linear group generated by reflections. Then one of the following holds:*

(a) $G = RL_n(q)$ *or* $G = RU_n(q)$, *where* $p \mid q$,
(b) ${\Omega_n}^{(\pm)}(q) < G \leq {GO_n}^{(\pm)}(q)$, $G \neq {SO_n}^{(\pm)}(q)$, *where* $p \mid q$,
(c) $G = \mathfrak{S}_{n+1}$, $p \nmid (n+1)$,
(d) $G = \mathfrak{S}_{n+2}$, $p \mid (n+2)$, $n \geq 5$,
(e) $G = W_p(G_i)$, $23 \leq i \leq 37$, $i \neq 25, 26, 32$, *and either* $p \in \{3, 5, 7\}$ *corresponds to one of the columns marked* p *or* np *in [5, Table 6.3], or* $p \nmid |G|$,

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