Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

The isomorphism problem for quantum affine spaces, homogenized quantized Weyl algebras, and quantum matrix algebras

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ARTICLE INFO

Article history:
Received 2 June 2016
Received in revised form 1 December 2016
Available online xxxx
Communicated by S. Iyengar

ABSTRACT

Bell and Zhang have shown that if A and B are two connected graded algebras finitely generated in degree one that are isomorphic as ungraded algebras, then they are isomorphic as graded algebras. We exploit this result to solve the isomorphism problem in the cases of quantum affine spaces, quantum matrix algebras, and homogenized multiparameter quantized Weyl algebras. Our result involves determining the degree one normal elements, factoring out, and then repeating. This creates an iterative process that allows one to determine relationships between relative parameters.

1. Introduction

Throughout, $\Bbbk$ is a field and all algebras are $\Bbbk$-algebras. All isomorphisms should be read as 'isomorphisms as $\Bbbk$-algebras'. Our primary source for all definitions is [4].

Hypothesis 1. A is a connected graded algebra finitely generated over $\Bbbk$ in degree 1.

Let R and S be algebras satisfying Hypothesis 1 with bases $\{x_i\}$ and $\{y_i\}$, respectively. Then R and S are isomorphic as graded algebras if there exists an algebra isomorphism $\Phi : R \to S$ such that $\Phi(x_i) = \sum \alpha_{ij} y_j$, $\alpha_{ij} \in \Bbbk$, for each x_i. If $a_{ij} \neq 0$, then we say y_j is a summand of $\Phi(x_i)$. Because of the following result, we will often assume without comment that isomorphisms between graded algebras are graded isomorphisms.

Theorem 1 *(Bell, Zhang [3, Theorem 0.1]). Let A and B be two algebras satisfying Hypothesis 1. If $A \cong B$ as ungraded algebras, then $A \cong B$ as graded algebras.*

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http://dx.doi.org/10.1016/j.jpaa.2016.12.036

A square matrix $\mathbf{p} = (p_{ij}) \in \mathcal{M}_n(\Bbbk^\times)$ is multiplicatively antisymmetric if $p_{ii} = 1$ and $p_{ij} = p_{ji}^{-1}$ for all $i \neq j$. Let $\mathcal{A}_n \subset \mathcal{M}_n(\Bbbk^\times)$ be the subset of multiplicatively antisymmetric matrices. A matrix $\mathbf{q} \in \mathcal{A}_n$ is a permutation of $\mathbf{p}$ if there exists a permutation $\sigma \in \mathcal{S}_n$ such that $q_{ij} = p_{\sigma(i)\sigma(j)}$ for all i, j.

For $\mathbf{p} \in \mathcal{A}_n$, the (multiparameter) quantum affine n-space $\mathcal{O}_{\mathbf{p}}(\Bbbk^n)$ is defined as the algebra with basis $\{x_i\}$, $1 \leq i \leq n$, subject to the relations $x_i x_j = p_{ij} x_j x_i$ for all $1 \leq i, j \leq n$.

Theorem 2 *(Theorem 2.4). $\mathcal{O}_{\mathbf{p}}(\Bbbk^n) \cong \mathcal{O}_{\mathbf{q}}(\Bbbk^m)$ if and only if $m = n$ and $\mathbf{p}$ is a permutation of $\mathbf{q}$.*

Mori proved Theorem 2.4 when $n = 3$ [10, Example 4.10] and this was extended to all n by Vitoria [11, Lemma 2.3]. We present a simple, self-contained proof that does not rely on the noncommutative projective algebraic geometry associated to $\mathcal{O}_{\mathbf{p}}(\Bbbk^n)$.

(Multiparameter) quantized Weyl algebras may be regarded as γ-difference operators on $\mathcal{O}_{\mathbf{p}}(\Bbbk^n)$. Let $\mathbf{p} \in \mathcal{A}_n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\Bbbk^\times)^n$. Then $A_n^{\mathbf{p},\gamma}(\Bbbk)$ is the algebra with basis $\{x_i, y_i\}$, $1 \leq i \leq n$, subject to the relations

$$\begin{aligned}
y_i y_j &= p_{ij} y_j y_i && \text{(all } i, j) \\
x_i x_j &= \gamma_i p_{ij} x_j x_i && (i < j) \\
x_i y_j &= p_{ji} y_j x_i && (i < j) \\
x_i y_j &= \gamma_j p_{ji} y_j x_i && (i > j) \\
x_j y_j &= 1 + \gamma_j y_j x_j + \sum_{l<j} (\gamma_l - 1) y_l x_l && \text{(all } j).
\end{aligned}$$

In the case of $n = 1$, the parameter $\mathbf{p}$ plays no role and so we refer to the single parameter simply as p and write $A_1^p(\Bbbk)$. By [5, Section 6], $A_1^p(\Bbbk) \cong A_1^q(\Bbbk)$ if and only if $p = q^{\pm 1}$. Goodearl and Hartwig solved the isomorphism problem for quantized Weyl algebras when no γ_i is a root of unity [7, Theorem 5.1]. They proved that if $A_n^{\mathbf{p},\gamma}(\Bbbk) \cong A_m^{\mathbf{q},\mu}(\Bbbk)$, then $n = m$ and there exists a sign vector $\varepsilon \in \{\pm 1\}^n$ such that

$$\mu_i = \gamma_i^{\varepsilon_i} \tag{1-1}$$

and $\mathbf{p}, \mathbf{q}$ satisfy

$$q_{ij} = \begin{cases} p_{ij} & \text{if } (\varepsilon_i, \varepsilon_j) = (1, 1), \\ p_{ji} & \text{if } (\varepsilon_i, \varepsilon_j) = (-1, 1), \\ \gamma_i^{-1} p_{ji} & \text{if } (\varepsilon_i, \varepsilon_j) = (1, -1), \\ \gamma_i p_{ij} & \text{if } (\varepsilon_i, \varepsilon_j) = (-1, -1). \end{cases} \tag{1-2}$$

The isomorphisms they give hold regardless of the root-of-unity condition. Levitt and Yakimov have extended this result to the case where all parameters are roots of unity and $A_n^{\mathbf{p},\gamma}(\Bbbk), A_m^{\mathbf{q},\mu}(\Bbbk)$ are free over their centers by utilizing noncommutative discriminants [9, Corollary 6.4]. Several intermediate cases are still open.

The quantized Weyl algebras are not graded and so we consider their homogenizations. The homogenized (multiparameter) quantized Weyl algebra $H_n^{\mathbf{p},\gamma}$ has algebra basis $\{z, x_i, y_i\}$, $1 \leq i \leq n$, where z commutes with the x_i and y_i. The relations in $H_n^{\mathbf{p},\gamma}$ are the same as those for $A_n^{\mathbf{p},\gamma}(\Bbbk)$ except the final relation type is replaced by its homogenization,

$$x_j y_j = z^2 + \gamma_j y_j x_j + \sum_{l<j} (\gamma_l - 1) y_l x_l.$$