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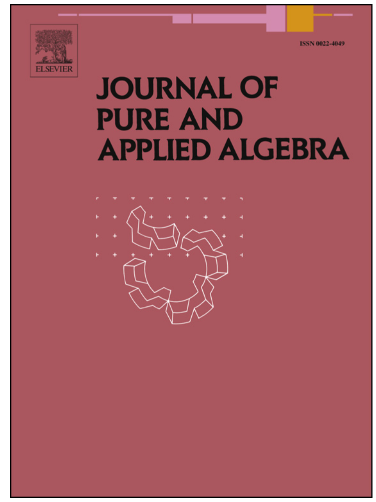

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Annihilator-stability and Unique Generation

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Abstract

A ring R is said to be left uniquely generated if $Ra = Rb$ in R implies that $a = ub$ for some unit u in R. These rings have been of interest since Kaplansky introduced them in 1949 in his classic study of elementary divisors. Writing $\mathtt{l}(b) = \{r \in R \mid rb = 0\}$, a theorem of Canfell asserts that R is left uniquely generated if and only if, whenever $Ra + \mathtt{l}(b) = R$ where $a, b \in R$, then $a - u \in \mathtt{l}(b)$ for some unit u in R. By analogy with the stable range 1 condition we call a ring with this property left annihilator-stable. In this paper we exploit this perspective on the left UG rings to construct new examples and derive new results. For example, writing J for the Jacobson radical, we show that a semiregular ring R is left annihilator-stable if and only if R/J is unit-regular, an analogue of Bass' theorem that semilocal rings have stable range 1.

Key Words: Stable range, annihilator-stable rings, uniquely generated rings, von Neumann regular rings, unit-regular rings, clean rings, triangular matrix rings, pseudo-morphic rings
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Throughout this paper every ring R is associative with unity, all modules are unitary, and R-morphisms are written opposite to the scalars. We write $J(R)$ for the Jacobson radical, $U(R)$ for the unit group, and $Z_l(R)$ and $Z_r(R)$ for the left and right singular ideals of R. We write $I \lhd R$ when I is an ideal of R. For modules we write $N \subseteq^{ess} M$ to mean that N is essential in M. The ring of $n \times n$ matrices over R will be denoted $M_n(R)$. The left and right annihilators of a set X are denoted $\mathtt{l}(X)$ and $\mathtt{r}(X)$ respectively. We denote the ring of integers by $\mathbb{Z}$ and write $\mathbb{Z}_n$ for the ring of integers modulo n. The term "regular ring" means von Neumann regular ring. Given a ring-theoretic condition $\mathfrak{c}$, a ring will be called a $\mathfrak{c}$-ring if it is both a left $\mathfrak{c}$-ring and a right $\mathfrak{c}$-ring, with a similar convention for elements.

1. Introduction

There are many important conditions on an element a in a ring R that demand the existence of a certain unit. Notable examples include:

$a \in R$ is left **uniquely generated** (**UG**) if $Ra = Rb$, $b \in R$, implies $a = ub$ for a unit u in R;

$a \in R$ has left **stable range 1 (SR1)** if $Ra + Rb = R$, $b \in R$, implies $a - u \in Rb$ for a unit u in R;

$a \in R$ is **unit-regular** if $aua = a$ for some unit u in R;

$a \in R$ is **clean** if $a = e + u$ where $e^2 = e$ and u is a unit in R.

In each case a ring R is said to have one of these conditions when **every element** of R has the condition. Here is some background about these classes of rings.