JPAA:5612

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Enveloping algebras that are principal ideal rings ☆

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A R T I C L E I N F O

Article history:
Received 28 June 2016
Received in revised form 29 December 2016
Available online xxxx
Communicated by E.M. Friedlander

MSC:
16S30; 17B50; 13F10

A B S T R A C T

Let L be a restricted Lie algebra over a field of positive characteristic. We prove that the restricted enveloping algebra of L is a principal ideal ring if and only if L is an extension of a finite-dimensional torus by a cyclic restricted Lie algebra.

1. Introduction

Let R be a ring with identity. Recall that R is called a *principal right ideal ring* (*pri-ring* for short) if every right ideal of R is principal. It is clear that such rings are right Noetherian and furthermore this property is inherited by homomorphic images. Similarly one defines *principal left ideal rings* (*pli-rings*). Although there are examples of pri-rings that are not pli-rings and vice versa (see e.g. [15], §1, Example 1.25), the two properties turn out to be equivalent provided R has an involution. For instance, this is the case when R is a group algebra or an (ordinary or restricted) enveloping algebra. If a ring is both a principal right and left ideal ring then we simply call it a *principal ideal ring*. It is known that a commutative principal ideal ring is a finite direct sum of rings, which are either integral domains or are completely primary. Ore [23] then was interested in certain differential polynomials which constitute a non-commutative principal ideal ring. This work was picked up and studied extensively by numerous algebraists including Asano [2], Amitsur [1], Jacobson [12], and others. Most notably, Goldie in [9] proved that a semiprime pri-ring is a finite direct sum of prime pri-rings and a prime pri-ring is a full matrix ring $M_n(K)$, where K is a right Noetherian integral domain. He further proved that a left Noetherian pri-ring is a finite direct sum of pri-primary rings.

☆ The research of the second author was supported by NSERC of Canada under grant # RGPIN 418201.
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http://dx.doi.org/10.1016/j.jpaa.2017.01.003
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Later Johnson in [14] dropped the Noetherian assumption in Goldie's Theorem and proved that a ring is a pri-ring if and only if it is a finite direct sum of primary pri-rings.

There has been also significant attention to group rings. First, Morita in [20] characterized those finite groups whose group algebras over an algebraically closed field are principal ideal rings. Fisher and Sehgal in [8] extended Morita's result to nilpotent groups over any field, and later Passman [24] dropped the nilpotence assumption from Fisher–Sehgal's result. We summarize the group ring result in the following. We denote the augmentation ideal of a group ring KG by $\omega(KG)$ and, for a prime p, a finite group is called a p'-group if its order is not divisible by p.

Theorem 1.1 *([8,24]). Let KG be the group ring of G over a field K. The following statements are equivalent.*

(1) *KG is a principal right ideal ring.*
(2) *KG is right Noetherian and $\omega(KG)$ is principal as a right ideal.*
(3) $\operatorname{char} K = 0$*: G is finite or finite-by-infinite cyclic.*
$\operatorname{char} K = p > 0$*: G is finite p'-by cyclic p or finite p'-by infinite cyclic.*

We just mention that among many other results, Farkas and Snider in [7] determined when the augmentation ideal of the group ring RG is principal as a right ideal, where R is a commutative integral domain of characteristic 0, and a characterization of semigroup algebras that are pri-rings was obtained by Jespers and Okniński in [13].

In this paper we settle the same problem for another important class of Hopf algebras. Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. The restricted universal enveloping algebra of L is denoted by $u(L)$. We will characterize L when $u(L)$ is a principal ideal ring. Before stating our main result, we recall that an abelian restricted Lie algebra T is called a torus if, for every $x \in T$, the restricted subalgebra generated by $x^{[p]}$ contains x. Our main result is as follows.

Theorem 1.2. *Let L be a restricted Lie algebra over a field of positive characteristic. Then $u(L)$ is a principal ideal ring if and only if L is an extension of a finite-dimensional torus by a cyclic restricted Lie algebra.*

In other words, $u(L)$ is a principal ideal ring if and only if there exists a finite-dimensional torus T such that T is an ideal of L and L/T is cyclic as a restricted Lie algebra. In particular, we deduce from Theorem 1.2 that if $u(L)$ is a principal ideal ring then L is abelian. We note that, by Theorem 1.1, this is not the case for group algebras, mainly due to the fact that semisimple group algebras are not necessarily commutative.

We briefly explain the strategy of the proof. The sufficiency part is not difficult. In order to prove the necessity, we first deal with the case that L is finite-dimensional in Theorem 3.4 and consider the restricted ideal T given by the intersection of L with the last power of the augmentation ideal of $u(L)$. We show that $u(T)$ is semisimple by proving that the counit of this Hopf algebra is not vanishing on the subspace of left integrals. By a well-known result of Hochschild [11], this allows us to conclude that T is a torus. Then the proof of the general case will use Theorem 3.4 in a crucial way.

We also show that the ordinary enveloping algebra $U(L)$ of an arbitrary Lie algebra L is a principal ideal ring if and only if L is either zero or 1-dimensional. In a more general setting, we think that finding the conditions under which an arbitrary Hopf algebra is a principal ideal ring would be an interesting future problem, and probably a difficult one.

2. Preliminaries

Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. The terms of the lower central series of L are defined by $\gamma_1(L) = L$ and $\gamma_{n+1}(L) = [\gamma_n(L), L]$, for every $n \geq 2$. We write L' for $\gamma_2(L)$ and