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Classical limits of quantum toroidal and affine Yangian algebras

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A B S T R A C T

In this short article, we compute the *classical limits* of the quantum toroidal and affine Yangian algebras of $\mathfrak{sl}_n$ by generalizing our arguments for $\mathfrak{gl}_1$ from [7] (an alternative proof for $n > 2$ is given in [10]). We also discuss some consequences of these results.

0. Introduction

The primary purpose of this note is to provide proofs for the description of the *classical limits* of the algebras $\mathcal{U}^{(n)}_{q,d}$ and $\mathcal{Y}^{(n)}_{h,\beta}$ from [4,9]. Here $\mathcal{U}^{(n)}_{q,d}$ and $\mathcal{Y}^{(n)}_{h,\beta}$ are the quantum toroidal and the affine Yangian algebras of $\mathfrak{sl}_n$ (if $n \geq 2$) or $\mathfrak{gl}_1$ (if $n = 1$), while *classical limits* refer to the limits of these algebras as $q \to 1$ or $h \to 0$, respectively. We also discuss the *classical limits* of certain constructions for $\mathcal{U}^{(n)}_{q,d}$.

The case $n = 1$ has been essentially worked out in [7]. In this note, we follow the same approach to prove the $n > 1$ generalizations. While writing down this note, we found that the $n \geq 3$ case has been considered in [10] long time ago (to deduce our Theorems 2.1 and 2.2, one needs to combine [10] with [1]). Hence, the only essentially new case is $n = 2$. Meanwhile, we expect our direct arguments to be applicable in some other situations of interest.

This paper is organized as follows:

• In Section 1, we recall explicit definitions of the Lie algebras $\ddot{u}^{(n)}_d$ and $\ddot{y}^{(n)}_\beta$, whose universal enveloping algebras coincide with the *classical limits* of $\mathcal{U}^{(n)}_{q,d}$ and $\mathcal{Y}^{(n)}_{h,\beta}$. We also recall the notion of $n\times n$ matrix algebras over the algebras of difference/differential operators on $\mathbb{C}^\times$ and their central extensions, denoted by $\bar{\mathfrak{d}}^{(n)}_t$ and $\bar{\mathfrak{D}}^{(n)}_s$, respectively.

• In Section 2, we establish two key isomorphisms relating the *classical limit* Lie algebras $\ddot{u}^{(n)}_d, \ddot{y}^{(n)}_\beta$ to the aforementioned Lie algebras $\bar{\mathfrak{d}}^{(n)}_{d^n}, \bar{\mathfrak{D}}^{(n)}_{n\beta}$.

• In Section 3, we discuss the *classical limits* of the following constructions for $\mathcal{U}^{(n)}_{q,d}$ $(n \geq 2)$:

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– the *vertical* and *horizontal* copies of a quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$ inside $\mathcal{U}^{(n)}_{q,d}$ from [3],
– the Miki's automorphism $\varpi : \mathcal{U}^{(n)}_{q,d} \xrightarrow{\sim} \mathcal{U}^{(n)}_{q,d}$ from [5],
– the commutative subalgebras $\mathcal{A}(s_0, \ldots, s_{n-1})$ of $\mathcal{U}^{(n),+}_{q,d}$ from [4].

1. Basic constructions

1.1. The quantum toroidal algebra $\mathcal{U}^{(n)}_{q,d}$ and the affine Yangian $\mathcal{Y}^{(n)}_{h,\beta}$

For $n \in \mathbb{N}$, set $[n] := \{0, 1, \ldots, n-1\}$ viewed as a set of mod n residues and $[n]^\times := [n]\backslash\{0\}$. For $n \geq 2$, we set $a_{i,j} := 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$ and $m_{i,j} := \delta_{i,j+1} - \delta_{i,j-1}$ for all $i, j \in [n]$.

∘ Given $h, \beta \in \mathbb{C}$, let $\mathcal{Y}^{(n)}_{h,\beta}$ be the affine Yangian of $\mathfrak{sl}_n$ (if $n \geq 2$) or $\mathfrak{gl}_1$ (if $n = 1$) as considered in [9], where it was denoted by $\mathcal{Y}^{(n)}_{\beta-h,2h,-\beta-h}$. These are unital associative $\mathbb{C}$-algebras generated by $\{x^\pm_{i,r}, \xi_{i,r}\}^{r\in\mathbb{Z}_+}_{i\in[n]}$ (here $\mathbb{Z}_+ := \{s \in \mathbb{Z} \mid s \geq 0\} = \mathbb{N} \cup \{0\}$) and with the defining relations as in [9, Sect. 1.2]. We will list these relations only for $h = 0$, which is of main interest in the current paper.

∘ Given $q, d \in \mathbb{C}^\times$, let $\mathcal{U}^{(n)}_{q,d}$ be the quantum toroidal algebra of $\mathfrak{sl}_n$ (if $n \geq 2$) or $\mathfrak{gl}_1$ (if $n = 1$) as considered in [4] but without the generators $q^{\pm d_1}, q^{\pm d_2}$ and with $\gamma^{\pm 1/2} = q^{\pm c/2}$. These are unital associative $\mathbb{C}$-algebras generated by $\{e_{i,k}, f_{i,k}, h_{i,k}, c\}^{k\in\mathbb{Z}}_{i\in[n]}$ and with the defining relations specified in [4, Sect. 2.1 and 5]. We note that algebras $\mathcal{U}^{(n)}_{\frac{d}{q},q^2,\frac{1}{dq}}$ from [9, Sect. 1.1] are their central quotients.

1.2. The Lie algebra $\ddot{u}^{(n)}_d$

In the $q \to 1$ limit, all the defining relations of $\mathcal{U}^{(n)}_{q,d}$ become of *Lie type*. Therefore, the $q \to 1$ limit of $\mathcal{U}^{(n)}_{q,d}$ is isomorphic to the universal enveloping algebra $U(\ddot{u}^{(n)}_d)$. The Lie algebra $\ddot{u}^{(n)}_d$ is generated by $\{\bar{e}_{i,k}, \bar{f}_{i,k}, \bar{h}_{i,k}, \bar{c}\}^{k\in\mathbb{Z}}_{i\in[n]}$ with $\bar{c}$ being a central element and the rest of the defining relations (u1–u7.2) to be given below in each of the 3 cases of interest: $n > 2$, $n = 2$, and $n = 1$.

• For $n > 2$, the defining relations are

$$[\bar{h}_{i,k}, \bar{h}_{j,l}] = k a_{i,j} d^{-km_{i,j}} \delta_{k,-l} \bar{c}, \tag{u1}$$

$$[\bar{e}_{i,k+1}, \bar{e}_{j,l}] = d^{-m_{i,j}} [\bar{e}_{i,k}, \bar{e}_{j,l+1}], \tag{u2}$$

$$[\bar{f}_{i,k+1}, \bar{f}_{j,l}] = d^{-m_{i,j}} [\bar{f}_{i,k}, \bar{f}_{j,l+1}], \tag{u3}$$

$$[\bar{e}_{i,k}, \bar{f}_{j,l}] = \delta_{i,j} \bar{h}_{i,k+l} + k\delta_{i,j}\delta_{k,-l}\bar{c}, \tag{u4}$$

$$[\bar{h}_{i,k}, \bar{e}_{j,l}] = a_{i,j} d^{-km_{i,j}} \bar{e}_{j,l+k}, \tag{u5}$$

$$[\bar{h}_{i,k}, \bar{f}_{j,l}] = -a_{i,j} d^{-km_{i,j}} \bar{f}_{j,l+k}, \tag{u6}$$

$$\sum_{\pi\in\Sigma_2} [\bar{e}_{i,k_{\pi(1)}}, [\bar{e}_{i,k_{\pi(2)}}, \bar{e}_{i\pm1,l}]] = 0 \text{ and } [\bar{e}_{i,k}, \bar{e}_{j,l}] = 0 \text{ for } j \neq i, i \pm 1, \tag{u7.1}$$

$$\sum_{\pi\in\Sigma_2} [\bar{f}_{i,k_{\pi(1)}}, [\bar{f}_{i,k_{\pi(2)}}, \bar{f}_{i\pm1,l}]] = 0 \text{ and } [\bar{f}_{i,k}, \bar{f}_{j,l}] = 0 \text{ for } j \neq i, i \pm 1. \tag{u7.2}$$

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