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Existence of unimodular elements in a projective module

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ABSTRACT

(1) Let R be an affine algebra over an algebraically closed field of characteristic 0 with $\dim(R) = n$. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n with determinant L. Suppose I is an ideal of A of height n such that there are two surjections $\alpha : P \twoheadrightarrow I$ and $\phi : L \oplus A^{n-1} \twoheadrightarrow I$. Assume that either (a) $k = 1$ and $n \geq 3$ or (b) k is arbitrary but $n \geq 4$ is even. Then P has a unimodular element (see 4.1, 4.3).

(2) Let R be a ring containing $\mathbb{Q}$ of even dimension n with height of the Jacobson radical of $R \geq 2$. Let P be a projective $R[T, T^{-1}]$-module of rank n with trivial determinant. Assume that there exists a surjection $\alpha : P \twoheadrightarrow I$, where $I \subset R[T, T^{-1}]$ is an ideal of height n such that I is generated by n elements. Then P has a unimodular element (see 3.4).

1. Introduction

Let R be a commutative Noetherian ring of dimension n. A classical result of Serre [22] asserts that if P is a projective R-module of rank $> n$, then P has a unimodular element. However, as is shown by the example of projective module corresponding to the tangent bundle of an even dimensional real sphere, this result is best possible in general. Therefore, it is natural to ask under what conditions P has a unimodular element when $rank(P) = n$. In [20], Raja Sridharan asked the following question.

Question 1.1. *Let R be a ring of dimension n and P be a projective R-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where $I \subset R$ is an ideal of height n such that I is generated by n elements. Does P have a unimodular element?*

Raja Sridharan proved that the answer to Question 1.1 is affirmative in certain cases (see [20, Theorems 3, 5]) and “negative” in general.

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Plumstead [17] generalized Serre's result and proved that if P is a projective $R[T]$-module of rank $> n$, then P has a unimodular element. Bhatwadekar and Roy [5] extended Plumstead's result and proved that projective $R[T_1, \dots, T_r]$-modules of rank $> n$ have a unimodular element. Mandal [16] proved analogue of Plumstead that projective $R[T, T^{-1}]$-modules of rank $> n$ have a unimodular element. In another direction, Bhatwadekar and Roy [4] proved that projective modules over $D = R[T_1, T_2]/(T_1T_2)$ of rank $> n$ have a unimodular element. Later Wiemers [26] extended this result and proved that if $D = R[T_1, \dots, T_r]/\mathcal{I}$ is a discrete Hodge algebra over R (here $\mathcal{I}$ is a monomial ideal), then projective D-modules of rank $> n$ have a unimodular element.

In view of results mentioned above, it is natural to ask the following question. Let A be either a polynomial ring over R or a Laurent polynomial ring over R or a discrete Hodge algebra over R. Let P be a projective A-module of rank n. Under what conditions P has a unimodular element? We will mention two such results.

Bhatwadekar and Raja Sridharan [7, Theorem 3.4] proved: *Let R be a ring of dimension n containing an infinite field. Let P be a projective $R[T]$-module of rank n. Assume P_f has a unimodular element for some monic polynomial $f \in R[T]$. Then P has a unimodular element.*

Das and Raja Sridharan [11, Theorem 3.4] proved: *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Let P be a projective $R[T]$-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of $R[T]$ of height n such that I is generated by n elements. Assume further that P/TP has a unimodular element. Then P has a unimodular element.*

Note that when n is odd, the above result is not known. Further, the requirement in the hypothesis that P/TP has a unimodular element is indeed necessary, in view of negative answer of Question 1.1. Motivated by Bhatwadekar–Sridharan and Das–Sridharan, we prove the following results.

Theorem 1.2. *(See 3.1) Let R be a ring of dimension n containing an infinite field. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n. Assume $P_{f(T_k)}$ has a unimodular element for some monic polynomial $f(T_k) \in A$. Then P has a unimodular element.*

Theorem 1.3. *(See 3.3) Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Let P be a projective $A = R[T_1, \cdots, T_k]$-module of rank n with determinant L. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of A of height n such that I is a surjective image of $L \oplus A^{n-1}$. Further assume that $P/(T_1, \cdots, T_k)P$ has a unimodular element. Then P has a unimodular element.*

Theorem 1.4. *(See 3.4) Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$. Assume that height of the Jacobson radical of R is ≥ 2. Let P be a projective $R[T, T^{-1}]$-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of $R[T, T^{-1}]$ of height n such that I is generated by n elements. Then P has a unimodular element.*

Theorem 1.5. *(See 3.6) Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$ and P be a projective $D = R[T_1, T_2]/(T_1T_2)$-module of rank n with determinant L. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where I is an ideal of D of height n such that I is a surjective image of $L \oplus D^{n-1}$. Further, assume that $P/(T_1, T_2)P$ has a unimodular element. Then P has a unimodular element.*

In view of above results, we end this section with the following question.

Question 1.6. *Let R be a ring of* **even** *dimension n containing $\mathbb{Q}$.*

(1) *Let P be a projective $A = R[T, T^{-1}]$-module of rank n with trivial determinant. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$, where $I \subset A$ is an ideal of height n such that I is generated by n elements. Assume further that $P/(T-1)P$ has a unimodular element. Does P has a unimodular element?*