Accepted Manuscript

A remark on regularity of powers and products of ideals

Winfried Bruns, Aldo Conca

PII: S0022-4049(17)30030-0
DOI: http://dx.doi.org/10.1016/j.jpaa.2017.02.005
Reference: JPAA 5623

To appear in: *Journal of Pure and Applied Algebra*

Received date: 7 October 2016
Revised date: 16 January 2017

A REMARK ON REGULARITY OF POWERS AND PRODUCTS OF IDEALS

WINFRIED BRUNS AND ALDO CONCA

To the memory of our friend Tony Geramita

ABSTRACT. We give a simple proof for the fact that the Castelnuovo-Mumford regularity and related invariants of products of powers of ideals are asymptotically linear in the exponents, provided that each ideal is generated by elements of constant degree. We provide examples showing that the asymptotic linearity is false in general. On the other hand, the regularity is always given by the maximum of finitely many linear functions whose coefficients belong to the set of the degrees of generators of the ideals.

1. INTRODUCTION

Let I be an homogeneous ideal of a polynomial ring $R = K[x_1,\dots,x_n]$. Cutkosky, Herzog and Trung [7] and, independently, Kodiyalam [10] proved in that the Castelnuovo-Mumford regularity $\operatorname{reg}(I^a)$ of the powers of I is a linear function of $a \in \mathbb{N}$ for large a. In [7, Remark pg.252] it is asserted that the same result holds for products of powers of ideals $I_1,\dots,I_m$, i.e., $\operatorname{reg}(I_1^{a_1}\cdots I_m^{a_m})$ is a linear function in $a = (a_1,\dots,a_m) \in \mathbb{N}^m$ if $a_i \gg 0$ for every i. We show that this is actually the case when each ideal I_i is generated by polynomials of a given degree, but false in general. The method of proof allows us to easily generalize the results to ideals in arbitrary standard graded K-algebras.

We refer the reader to [5] for basic commutative algebra.

After the fist version of this note had been uploaded to arXiv.org, M. Chardin informed us that the main result is a special case of a theorem proved by Bagheri, Chardin and Há [2, Throrem 4.6]. We hope that our simple approach to the problem and the accompanying examples are nevertheless welcome.

2. REGULARITY FOR POWERS AND PRODUCTS

For a finitely generated graded non-zero R-module M and a nonnegative integer $j \le \operatorname{pd}_R(M)$ we denote the largest degree of a minimal generator of the j-th syzygy module by $t_j^R(M)$, and, by convention, set $t_j^R(M) = -\infty$ if $j > \operatorname{pd}_R(M)$. By definition one has $\operatorname{reg}_R(M) = \max\{t_j^R(M) - j : j \ge 0\}$.

Let $I_1,\dots,I_m$ be non-zero homogeneous ideals of R and let $f_{i1},\dots,f_{ig_i}$ be a minimal homogeneous generating system of I_i. Set $d_{ij} = \deg f_{ij}$ and

$$B = R[z_{ij} : 1 \le i \le m, 1 \le j \le g_i] = K[x_1,\dots,x_n,z_{ij} : 1 \le i \le m, 1 \le j \le g_i].$$

To simplify the exposition we set

$$I^a = I_1^{a_1}\cdots I_m^{a_m} \quad \text{for} \quad a \in \mathbb{N}^m,$$

2010 *Mathematics Subject Classification.* 13D02, 13D40.

Key words and phrases. asymptotic regularity, powers of ideals, multi-Rees algebra.