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ALGORITHMIC RECOGNITION OF INFINITE CYCLIC EXTENSIONS

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Abstract. We prove that one cannot algorithmically decide whether a finitely presented $\mathbb{Z}$-extension admits a finitely generated base group, and we use this fact to prove the undecidability of the BNS invariant. Furthermore, we show the equivalence between the isomorphism problem within the subclass of unique $\mathbb{Z}$-extensions, and the semi-conjugacy problem for deranged outer automorphisms.

In the present paper, we study algorithmic problems about recognition of certain algebraic properties among some families of group extensions. Indeed, we see that yet for the relatively easy family of $\mathbb{Z}$-extensions one can find positive and negative results, i.e., both solvable and unsolvable "recognition problems".

For example, we prove that one cannot algorithmically decide whether a finitely presented $\mathbb{Z}$-extension admits a finitely generated base group. Even when the extension has a unique possible base group, it is not decidable in general whether this particular base group is finitely generated or not. As a consequence, we prove general undecidability for the Bieri–Neumann–Strebel invariant: there is no algorithm which, on input a finite presentation of a group G and a character $\chi\colon G \to \mathbb{R}$, decides whether $[\chi]$ belongs to the BNS invariant of G, $[\chi] \in \Sigma(G)$, or not. Although this result seems quite natural, since this geometric invariant has long been agreed to be hard to compute in general (see for example [17, 18, 25, 28]), as far as we know, its undecidability does not seem to be contained in the literature. Following our study of recognition properties, we finally consider the isomorphism problem in certain classes of unique $\mathbb{Z}$-extensions, and prove that it is equivalent to the semi-conjugacy problem for the corresponding deranged outer automorphisms (see details in Section 8).

The structure of the paper is as follows. In Section 1 we state the recognition problems we are interested in. In Section 2 we introduce the most general framework for our study: (finitely presented) $\mathbb{Z}^r$-extensions (denoted $*$-by-$\mathbb{Z}^r$), unique $\mathbb{Z}^r$-extensions (denoted $!$-by-$\mathbb{Z}^r$), as well as the subfamily of fg-by-$\mathbb{Z}^r$ groups, and will investigate the above problems for them. In Sections 3 and 4 we focus on the case $r = 1$ (i.e., infinite cyclic extensions) which will be the main target of the paper. The central result in Section 5 is Theorem 5.3, showing that the membership problem for fg-by-$\mathbb{Z}$ (among other similar families) is undecidable, even within the class $!$-by-$\mathbb{Z}$. As an application, Section 6 contains the undecidability of the BNS invariant (Theorem 6.4). In Section 7 we search for "standard presentations" of fg-by-$\mathbb{Z}$ groups (Proposition 7.1). Finally, in Section 8 we characterize the isomorphism problem in the subclass of unique $\mathbb{Z}$-extensions by means of the so-called semi-conjugacy problem (a weakened version of the standard conjugacy problem) for deranged outer automorphisms (Theorem 8.12).

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