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THE NASH-MOSER THEOREM OF HAMILTON AND RIGIDITY OF FINITE DIMENSIONAL NILPOTENT LIE ALGEBRAS

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Abstract. We apply the Nash-Moser theorem for exact sequences of R. Hamilton to the context of deformations of Lie algebras and we discuss some aspects of the scope of this theorem in connection with the polynomial ideal associated to the variety of nilpotent Lie algebras. This allows us to introduce the space $H^2_{k-nil}(\mathfrak{g},\mathfrak{g})$, and certain subspaces of it, that provide fine information about the deformations of $\mathfrak{g}$ in the variety of k-step nilpotent Lie algebras.

Then we focus on degenerations and rigidity in the variety of k-step nilpotent Lie algebras of dimension n with $n \leq 7$ and, in particular, we obtain rigid Lie algebras and rigid curves in the variety of 3-step nilpotent Lie algebras of dimension 7. We also recover some known results and point out a possible error in a published article related to this subject.

1. Introduction

In this paper we will assume that all Lie algebras and representations are finite dimensional, and mostly over $\mathbb{R}$. Here we apply a finite dimensional version of the Nash-Moser theorem for exact sequences of R. Hamilton to the context of deformations in the variety of nilpotent Lie algebras. Our main results are described below.

1.1. The Nash-Moser theorem of R. Hamilton.

A very well known general principle of deformation theory says that given an (algebraic) structure μ, then

$$H^2(\mu,\mu) = 0 \;\Rightarrow\; \mu \text{ is rigid, but the converse is not true in general.} \tag{1.1}$$

By definition, an algebraic structure μ on a $\mathbb{K}$-vector space V is rigid if the $GL(V)$-orbit of μ, $\mathcal{O}(\mu)$, is a Zariski open set in the algebraic variety of all such algebraic structures.

Roughly speaking, when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, an algebraic structure μ is rigid if every small perturbation of μ is isomorphic to μ. More precisely, it is known that $\mathcal{O}(\mu)$ is open in the metric topology if and only if it is open in the Zariski topology (see [NR, Proposition 17.1], see also [GK, Proposition 2]). As a consequence of this, the principle (1.1) follows from a particular instance of the Nash-Moser theorem for exact sequences of R. Hamilton as we recall below. This theorem is stated in [H] in terms of tame Fréchet spaces and it is related to the inverse function theorem of Nash and Moser

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