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A criterion for an abelian variety to be non-simple ☆

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ABSTRACT

We give a criterion in terms of period matrices for an arbitrary polarized abelian variety to be non-simple. Several examples are worked out.

1. Introduction

Let $(A, \mathcal{L})$ be a complex abelian variety of dimension g with polarization of type $D = \operatorname{diag}(d_1, \ldots, d_g)$ defined by an ample line bundle $\mathcal{L}$. So $A = V/\Lambda$ where V is a complex vector space of dimension g and Λ is a lattice of maximal rank in $\mathbb{C}^g$ such that with respect to a basis of V and a symplectic basis of Λ, A is given by a period matrix $(D\ Z)$ with Z in the Siegel upper half space of rank g. The aim of this paper to give a set of equations in the entries of the matrix Z which characterize the fact that $(A, \mathcal{L})$ is non-simple. These equations are easy to work out for $g = 2$ and can be given explicitly with the help of a computer program for $g = 3$.

To be more precise, the polarization induces a bijection

$$\varphi : \mathrm{NS}_{\mathbb{Q}}(A) \to \mathrm{End}^s_{\mathbb{Q}}(A)$$

of the rational Néron–Severi group $\mathrm{NS}_{\mathbb{Q}}(A) := \mathrm{NS}(A) \otimes \mathbb{Q} = \left(\mathrm{Pic}(A)/\mathrm{Pic}^0(A)\right) \otimes \mathbb{Q}$ with the $\mathbb{Q}$-vector space $\mathrm{End}^s_{\mathbb{Q}}(A) := \mathrm{End}^s(A) \otimes \mathbb{Q}$ generated by the endomorphisms of A which are symmetric with respect to the Rosati involution of $(A, \mathcal{L})$. Now an abelian subvariety X of A corresponds to a symmetric idempotent $\varepsilon_X \in \mathrm{End}^s_{\mathbb{Q}}(A)$. So $\varphi^{-1}(\varepsilon_X)$ is an element of $\mathrm{NS}_{\mathbb{Q}}(A)$. On the other hand, $\mathrm{NS}_{\mathbb{Q}}(A)$ admits an intersection

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product which associates to g elements $\alpha_1, \dots, \alpha_g \in \mathrm{NS}_{\mathbb{Q}}(A)$ a rational number $(\alpha_1 \cdots \alpha_g)$. Theorem 3.3 is a criterion for an element $\alpha \in \mathrm{NS}_{\mathbb{Q}}(A)$ to be equal to $\varphi^{-1}\varepsilon_X$ for an abelian subvariety X of A in terms of the intersection numbers $(\alpha^r \cdot \mathcal{L}^{g-r})$.

Introducing coordinates of Λ as above and using the fact that

$$\mathrm{NS}_{\mathbb{Q}}(A) = H^{1,1}(A) \cap H^2(A, \mathbb{Q})$$

we translate the criterion into terms of differential forms which finally gives the above mentioned equations in Theorem 4.1 for the matrix Z. These have been outlined in [1] in the case of a principally polarized abelian variety. For our applications we need however the generalization to an arbitrary polarized abelian variety as we will explain now.

Let G be a finite group acting faithfully on the abelian variety A. Following [3, Section 13.6], this action induces a morphism ρ from the group algebra $\mathbb{Q}[G]$ to the rational endomorphism algebra $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\mathbb{Q}[G]$ is a semisimple algebra, it decomposes as a product of simple algebras $Q_1 \times \cdots \times Q_r$. Each Q_i is generated by a central idempotent e_i, and these are in correspondence with the rational irreducible representations of G. By defining $A_i = \rho(me_i)$, where m is an integer such that $\rho(me_i) \in \mathrm{End}(A)$, the so called isotypical decomposition of A is obtained. It is an isogeny $A_1 \times \cdots \times A_r \to A$ where the A_i are abelian subvarieties of A uniquely determined by the simple factors Q_i of the rational group algebra $\mathbb{Q}[G]$.

In an analogous way the factors A_i are decomposed further up to isogeny as $A_i \sim B_i^{n_i}$. This last decomposition for each isotypical factor comes from the decomposition of Q_i as a product of minimal ideals. Therefore here B_i is an abelian subvariety of A_i, not uniquely determined, and $n_i = \frac{\deg \chi_i}{m_i}$, where χ_i is a complex irreducible representation associated to the simple factor Q_i and m_i its Schur index (see [3, Section 13.6]). The decomposition

$$A \sim B_1^{n_1} \times \cdots \times B_r^{n_r} \tag{1.1}$$

is called the *group algebra decomposition* of the G-abelian variety A. Our starting point was the question whether the abelian varieties B_i are simple. Even if A is principally polarized, the induced polarization on B_i is in general not principal. So in order to discuss the simplicity of B_i we need Theorem 4.1 also in the non-principally polarized case. We will outline several examples for this.

In Section 2 we recall and outline some more details about the relation between abelian subvarieties and symmetric idempotents of a polarized abelian variety. Section 3 contains the above criterion in terms of the Néron–Severi group and Section 4 its translation in terms of period matrices. Finally Section 5 contains the examples.

List of Symbols

A	complex abelian variety
$\mathcal{L}$	ample line bundle on A
$(d_1, \dots, d_g)$	type of the line bundle $\mathcal{L}$
X	abelian subvariety of A
e_X	exponent of X
N_X	norm endomorphism of X
$'$	Rosati involution
$\mathrm{End}^s_{\mathbb{Q}}(A)$	$\mathbb{Q}$-endomorphisms of A fixed by $'$
$\mathrm{NS}_{\mathbb{Q}}(A)$	Néron–Severi group of A tensored with $\mathbb{Q}$
δ_X	numerical class associated to X in $\mathrm{NS}_{\mathbb{Q}}(A)$
α_X	$\frac{1}{e_X}\delta_X$
ε_X	symmetric idempotent associated to X