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An infinitesimal Noether–Lefschetz theorem for Chow groups

D. Patel [a], G.V. Ravindra [b,*]

[a] *Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA*
[b] *Department of Mathematics, University of Missouri, St. Louis, MO 63121, USA*

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ABSTRACT

Let X be a smooth, complex projective variety, and Y be a very general, sufficiently ample hypersurface in X. A conjecture of M.V. Nori states that the natural restriction map $\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(Y)_{\mathbb{Q}}$ is an isomorphism for all $p < \dim Y$ and an injection for $p = \dim Y$. This is *the generalized Noether–Lefschetz conjecture*. We prove an infinitesimal version of this conjecture.

1. Introduction

A fundamental theorem concerning the topology of algebraic varieties is the *Weak Lefschetz theorem* (also known as the *Lefschetz hyperplane section theorem*).

Theorem 1. *Let X be a smooth, projective variety of dimension $m+1$ over the field of complex numbers, and $Y \subset X$ be a hyperplane section. The restriction map of singular cohomology groups $\mathrm{H}^i(X,\mathbb{Z}) \to \mathrm{H}^i(Y,\mathbb{Z})$ is an isomorphism for $i < m$, and a monomorphism for $i = m$. Equivalently, one has that $\mathrm{H}^i(X,Y;\mathbb{Z}) = 0$ for $i \leq m$.*

The philosophy of motives, and the conjectures of Bloch and Beilinson imply (see e.g. §2, [13] for details) that motivic analogs of the above theorem should also be true, namely that

Conjecture 1 *(Weak Lefschetz conjecture). $\mathrm{CH}^p(X) \otimes \mathbb{Q} \to \mathrm{CH}^p(Y) \otimes \mathbb{Q}$ is an isomorphism for $p < m/2$, and a monomorphism for $p = m/2$.*

In the special case when $X = \mathbb{P}^{m+1}$, this conjecture is an old question of Hartshorne (see [7]). Very little is known about this conjecture, except in the case $p = 1$, where the statement even holds integrally. In this case, using the correspondence between divisors and line bundles, the theorem is usually stated as follows:

* Corresponding author.
E-mail addresses: patel471@purdue.edu (D. Patel), girivarur@umsl.edu (G.V. Ravindra).

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Theorem 2 *(Grothendieck–Lefschetz theorem, [6]). Let X be a smooth, projective variety of dimension at least 4, and Y be a smooth hyperplane section. The restriction map of Picard groups $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism.*

When X is a 3-fold, a slightly weaker result is true.

Theorem 3 *(Noether–Lefschetz theorem, [2]). Let X be a smooth, projective 3-fold, and Y be a very general, sufficiently ample hypersurface in X. The restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism.*

Pioneering work in the context of algebraic cycles, especially various refinements and extensions of the Noether–Lefschetz theorem, was carried out by M. Green and C. Voisin, among others, beginning in the 1980's (see [16] for a detailed account of these and related developments). Their results, especially [5] (which was also independently proved by C. Voisin (unpublished)), in turn inspired M. Nori to prove the following remarkable connectivity theorem (see Lemmas 2.1 and 2.2, and Theorem 4, [10]).

Theorem 4. *Let X be a smooth, projective variety of dimension $m+1$, and $\mathcal{O}_X(1)$ be a sufficiently ample line bundle. Let $S := |\mathcal{O}_X(1)|$, $A := X \times S$, and $B := \{(x, f) \in X \times S | f(x) = 0\}$ be the universal hypersurface. Then for any smooth morphism $g : T \to S$, one has $\mathrm{H}^p(A_T, \Omega^q_{(A_T,B_T)}) = 0$ for $p \leq m$ and $p + q \leq 2m$. Consequently, $\mathrm{H}^i(A_T, B_T; \mathbb{Q}) = 0$ for $i \leq 2m$.*

Here $\Omega^q_{(A_T,B_T)}$ is defined by the exact sequence

$$0 \to \Omega^q_{(A_T,B_T)} \to \Omega^q_{A_T} \to i_*\Omega^q_{B_T} \to 0,$$

where $i : B_T \to A_T$ is the natural inclusion. Let $k = k(S)$ denote the function field of the parameter space S above, and $\bar{k}$ denote its algebraic closure. Let $X_{\bar{k}} := X \times_{\mathbb{C}} \bar{k}$, and $Y := B \times_S \bar{k}$. We have the following consequence of Theorem 4.

Theorem 5. $\mathrm{H}^p(X_{\bar{k}}, \Omega^q_{(X_{\bar{k}},Y)}) = 0$, *for $p \leq m$ and $p + q \leq 2m$.*

Proof. The result follows from the fact that cohomology and Kähler differentials commute with direct limits. First, note that we can write $\bar{k}$ as the inverse limit of schemes T_α where each $T_\alpha \to S$ is finite étale over an affine open in S. Note that each T_α is affine (since it is finite over an open affine), and therefore the transition maps in the inverse system A_{T_α} are all affine. It follows that both $\varprojlim_{T_\alpha} A_{T_\alpha}$ and $\varprojlim_{T_\alpha} B_{T_\alpha}$ exist in the category of schemes. Since fiber products commute with taking inverse limits, one has $X_{\bar{k}} \cong \varprojlim_{T_\alpha} A_{T_\alpha}$ and $Y \cong \varprojlim_{T_\alpha} B_{T_\alpha}$. Moreover, the universal property of Kähler differentials implies that $\Omega^1_{X_{\bar{k}}} \cong \varinjlim_{T_\alpha} \Omega^1_{A_{T_\alpha}}$ and $\Omega^1_Y \cong \varinjlim_{T_\alpha} \Omega^1_{B_{T_\alpha}}$. Since taking exterior powers commutes with direct limits, we have an analogous result for the higher order Kähler differentials. One also has an analogous statement for the relative differentials, since taking direct limits is an exact functor. Combining everything we have:

$$\mathrm{H}^p(X_{\bar{k}}, \Omega^q_{(X_{\bar{k}},Y)}) \cong \mathrm{H}^p(\varprojlim_{T_\alpha} A_{T_\alpha}, \varinjlim_{T_\alpha} \Omega^q_{(A_{T_\alpha},B_{T_\alpha})}) \cong \varinjlim_{T_\alpha} \mathrm{H}^p(A_{T_\alpha}, \Omega^q_{(A_{T_\alpha},B_{T_\alpha})}).$$

Here the last isomorphism follows from the fact that cohomology commutes with direct limits. The result now follows from Theorem 4. $\square$

Using his connectivity theorem, Nori proved the existence of non-trivial cycles in the Griffiths group which are in fact not detected by the Abel–Jacobi map, thus generalizing the original result due to Griffiths.

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