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The nilpotent multipliers of crossed modules

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ABSTRACT

The article is devoted to extending the notions of the c-nilpotent multiplier and the c-stem cover of a group to the crossed modules context. We give a five-term exact sequence in the c-nilpotent multiplier associated with an extension of crossed modules from which we deduce a generalization version of the Basic Theorem of Grandjean and Ladra (1998) [9] relating nilpotent crossed modules to their second homologies. We also study c-stem extensions and c-covers of crossed modules and prove that any c-stem extension of a perfect crossed module is a homomorphic image of its c-stem cover. Finally, we show that the c-nilpotent multipliers of a perfect crossed module are isomorphic to each other. This generalizes the work of Burns and Ellis (1997) [5] in group theory.

1. Introduction

Let G be a group presented as the quotient of a free group F by a normal subgroup R. Then the *c-nilpotent multiplier* of G, $c \geq 1$, is defined to be the abelian group

$$\mathcal{M}^{(c)}(G) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F),$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-th term of the lower central series of F and $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$, inductively. The group $\mathcal{M}^{(1)}(G) = \mathcal{M}(G)$ is more usually known as the *Schur multiplier* of G (which is isomorphic to the second integral homology of G). In [2] Baer proved that $\mathcal{M}^{(c)}(G)$ is independent, up to isomorphism, of the chosen free presentation of G. This concept plays an important role in group theory and has since been further investigated by a number of authors ([3,5,6,8,12,14,15]). In particular, it is shown in [13] that any extension of groups induces a natural five-term exact sequence in the c-nilpotent multipliers. Furthermore, using the notion of non-abelian exterior product of groups, it was proved by Burns and Ellis [5] that for $c, d \geq 1$ and every perfect group G, $\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(d)}(G)$.

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In [7] Carrasco, Cegarra, and Grandjean extended the Barr-Beck cotriple homology of groups to crossed modules, based on a tripleable underlying functor to the category of sets. They, in addition, gave a Hopf formula for the second homology of a crossed module. Some important results concerning the second homology of crossed modules can be found in [7,10,13,19]. Vieites and Casas [20] presented the notions of stem extension and stem cover of crossed modules, and classified stem extensions of perfect crossed modules in terms of their second homologies. Also, it is proved in [17] that any crossed module has at least one stem cover and the structure of stem covers of crossed modules whose second homologies are finite is determined.

In this article, we introduce the concept of the c-nilpotent multiplier of a crossed module, which is a vast generalization of the c-nilpotent multiplier of groups and the second homology of crossed modules. Similarly to a result of Lue [14] in the group case, associated to an extension of crossed modules, we obtain a five-term exact sequence in the c-nilpotent multiplier. We also study two kinds of c-central extensions: namely c-stem extensions and c-stem covers, and give several results about them. Finally, we prove that the c-nilpotent multipliers of a perfect crossed module are isomorphic to the second homology.

2. Generalities on crossed modules

A *crossed module* (T, G, ∂) is a group homomorphism $\partial : T \longrightarrow G$ together with a group action $(g, t) \mapsto {}^g t$ of G on T satisfying $\partial({}^g t) = g\partial t g^{-1}$ and ${}^{\partial t} t' = tt't^{-1}$, for all $t, t' \in T$, $g \in G$. It is worth nothing that for any crossed module (T, G, ∂), the image of ∂ is a normal subgroup of G and its kernel is a G-invariant subgroup in the centre of T. Evidently, if N is a normal subgroup of a group G, then (N, G, i) is a crossed module, where i is the inclusion and G acts on N by conjugation. In this way, every group G can be seen as a crossed module in two obvious ways: $(1, G, i)$ or (G, G, id).

A *morphism of crossed modules* $(\alpha, \beta) : (T, G, \partial) \longrightarrow (M, P, \mu)$ is a pair of group homomorphisms $\alpha : T \longrightarrow M$ and $\beta : G \longrightarrow P$ such that $\mu\alpha = \beta\partial$ and α is a G-group homomorphism via β, i.e., $\alpha({}^g t) = {}^{\beta(g)}\alpha(t)$ for all $g \in G$, $t \in T$.

Taking objects and morphisms as defined above, we obtain the category $\mathfrak{CM}$ of crossed modules. In this category we have the notions of injection, surjection, (normal) subobject, kernel, cokernel, exact sequence, etc.; most of them can be found in detail in [13,18]. Furthermore, the category $\mathfrak{Gp}$ of groups can be regarded as a Birkhoff subcategory of $\mathfrak{CM}$ by means of the full embedding $\iota : \mathfrak{Gp} \longrightarrow \mathfrak{CM}$ which sends a group G to $\iota(G) = (1, G, i)$. The functor ι has a left adjoint $\tau : \mathfrak{CM} \longrightarrow \mathfrak{Gp}$, $\tau(T, G, \partial) = G/\partial(T)$, and a right adjoint $\kappa : \mathfrak{CM} \longrightarrow \mathfrak{Gp}$, $\kappa(T, G, \partial) = G$. Also, the other way of regarding a group G as a crossed module defines a functor $\varepsilon : \mathfrak{Gp} \longrightarrow \mathfrak{CM}$ given by $\varepsilon(G) = (G, G, id)$, which is the right adjoint to the functor κ and the left adjoint to the functor $\zeta : \mathfrak{CM} \longrightarrow \mathfrak{Gp}$ defined by $\zeta(T, G, \partial) = T$.

Let (T, G, ∂) be a crossed module with a normal crossed submodule (S, H, ∂). Then

(i) the *centre* of (T, G, ∂) is $Z(T, G, \partial) = (T^G, Z(G) \cap st_G(T), \partial)$, where $Z(G)$ is the centre of G, $T^G = \{t \in T | {}^g t = t \text{ for all } g \in G\}$ and $st_G(T) = \{g \in G | {}^g t = t \text{ for all } t \in T\}$. The crossed module (T, G, ∂) is called *abelian* if it coincides with its centre, or equivalently, if G is abelian and the action of G on T is trivial (which yields that T is also abelian). In the same way as in group theory, we now define the *upper central series* $Z_n(T, G, \partial)$, $n \geq 0$, of (T, G, ∂) inductively as follows:

$$Z_0(T, G, \partial) = 1,$$
$$\frac{Z_{n+1}(T, G, \partial)}{Z_n(T, G, \partial)} = Z\Big(\frac{(T, G, \partial)}{Z_n(T, G, \partial)}\Big) \quad \text{for} \quad n \geq 0.$$

Obviously, the terms $Z_n(T, G, \partial)$ are characteristic crossed submodules of (T, G, ∂).

(ii) The *commutator crossed module* $[(S, H, \partial), (T, G, \partial)]$ is defined as the normal crossed submodule $([G, S][H, T], [H, G], \partial)$ of (T, G, ∂), where $[G, S][H, T]$ denotes the normal subgroup of T generated by the

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