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# Finiteness conditions and cotorsion pairs

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A B S T R A C T

We study the interplay between the notions of $n$-coherent rings and finitely $n$-presented modules, and also study the relative homological algebra associated to them. We show that the $n$-coherency of a ring is equivalent to the thickness of the class of finitely $n$-presented modules. The relative homological algebra part comes from the study of orthogonal complements to this class of modules with respect to the Ext and Tor functors. We also construct cotorsion pairs from these orthogonal complements, allowing us to provide further characterizations of $n$-coherent rings.


## 0. Introduction

Finitely generated and finitely presented modules (over a ring $R$) are ubiquitous in homological algebra. Many module theoretical properties can be functorially described in terms of these two classes of modules. For example, using only finitely generated modules we can test whether a module is injective or not. We can also test whether a module $M$ is finitely presented by checking that the functor $\mathrm{Hom}_R(M, -)$ commutes with direct limits.

While finitely presented modules are finitely generated, the converse is not true in general. However, if $R$ is Noetherian, then these two classes of modules coincide. A first example of a (necessarily non-Noetherian) ring where these two classes of modules do not coincide is $k[x_1, x_2, x_3, \ldots]$, the ring of polynomials (over a field $k$) in countably infinite many variables. This is a well known coherent ring, that is, a ring over which we can find finitely generated modules that are not finitely presented. Recall that a finitely presented module is a finitely generated one such that it has a finite amount of relations between its finite collection of generators. Thus one can refine the class of finitely presented modules as modules that not only have these finiteness conditions (on generators and on relations between generators), but also have a finite amount of

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relations among the relations between the generators, and then a finite amount of relations among those relations of relations, and so on continuing up to the $n$th finite collection of relations among relations. This description gives rise to notion of *finitely $n$-presented modules*, which we denote by $\mathcal{FP}_n$. The study of the class of finitely $n$-presented modules is the content of Section 1, where some classic results are collected and examples exhibited.

These finiteness conditions over modules also motivate what can be thought as finiteness conditions over rings. The concept of Noetherian and coherent rings can be stated in terms of finitely generated and finitely presented modules, and therefore stated in terms of finitely $n$-presented modules. This observation allows us to generalize those (ring) definitions and naturally get to the notion of *$n$-coherent rings*. It is immediate then to ask about the connection between these two concepts: finitely $n$-presented modules and $n$-coherent rings. Section 2 deals with this question and, in particular, a characterization of $n$-coherent rings in terms of finitely $n$-presented modules is established.

We also study the relative homological algebra with respect to the class $\mathcal{FP}_n$, from the injective and flat perspective; that is, modules that have *a vanishing property* with respect to $\mathcal{FP}_n$, and the functors $\mathrm{Ext}^1_R(-,-)$ and $\mathrm{Tor}^R_1(-,-)$. These classes of modules are called *FP$_n$-injective modules* and FP$_n$-*flat modules*, respectively, and denoted $\mathcal{FP}_n$-Inj and $\mathcal{FP}_n$-Flat. Some of the presented results on this matter are adaptations of [4] and [7]. However, in the latter reference, the authors consider slightly different notions of relative injective modules and relative flat modules, while the former reference can be regarded as the $n = \infty$ case. This is done in Section 3, where most of the results are presented for the case $n > 1$, given that the cases $n = 0$ and $n = 1$ are well documented in the literature (see [12,14,17,18,21], for instance).

Section 4 studies the completeness of certain cotorsion pairs associated to the classes $\mathcal{FP}_n$-Inj and $\mathcal{FP}_n$-Flat. In the first half of this section, we study conditions under which $\mathcal{FP}_n$-Inj is the left and right half of two complete cotorsion pairs. The second half is about cotorsion pairs involving the class $\mathcal{FP}_n$-Flat. In the last section, we investigate conditions for when $R$ is an $n$-coherent ring in terms of the cotorsion pairs introduced in Section 4, and in particular, in terms of the classes $\mathcal{FP}_n$-Inj and $\mathcal{FP}_n$-Flat.

Throughout this paper, $R$ denotes an associative ring with unit, and $R$-Mod and Mod-$R$ the categories of left and right $R$-modules, respectively. All modules will be left $R$-modules in Sections 1, 2 and 4, and all the definitions and results stated will be also valid for right $R$-modules. In Sections 3 and 5, on the other hand, we will need to distinguish between left and right $R$-modules.

## 1. Finiteness conditions in modules

Let $n \geq 0$ be an integer. A left $R$-module $M$ is said to be **finitely $n$-presented**, if there is an exact sequence

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in $R$-Mod where the modules $F_i$ are finitely generated and free, for every $0 \leq i \leq n$. Such an exact sequence is called a **finite $n$-presentation** of $M$, and note that it is a truncated free resolution of $M$.

This way, whenever we are given a finitely $n$-presented module, then we may think of it as a finitely generated module, such that it has is a finite collection of relations between its generators, which in turn will have a finite amount of relations among those relations, and so on all the way up to $n$. The idea of finitely $n$-presented can be found in the literature, in particular in [3], and in [6] where they are referred to as *modules of type FP$_n$*.

Denote by $\mathcal{FP}_n$ the class of all finitely $n$-presented modules. Thus $\mathcal{FP}_0$ is the class of all finitely generated modules, and $\mathcal{FP}_1$ is the class of all finitely presented modules.

Hence, whenever a finite $n$-presentation of a module is exhibited, then we know that such module is in $\mathcal{FP}_n$; in turn, if we have a finite $k$-presentation of a module in $\mathcal{FP}_n$, with $k \leq n$, then we can extend that