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The local–global principle for symmetric determinantal representations of smooth plane curves in characteristic two

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ABSTRACT

We give an application of Mumford's theory of canonical theta characteristics to a Diophantine problem in characteristic two. We prove that a smooth plane curve over a global field of characteristic two is defined by the determinant of a symmetric matrix with entries in linear forms in three variables if and only if such a symmetric determinantal representation exists everywhere locally. It is a special feature in characteristic two because analogous results are not true in other characteristics.

1. Introduction

Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 1$ over a field K. The plane curve C is said to admit a *symmetric determinantal representation* over K if there is a symmetric matrix M of size d with entries in K-linear forms in three variables X, Y, Z such that C is defined by the equation $\det(M) = 0$. Two symmetric determinantal representations M, M' of C are said to be *equivalent* if $M' = \lambda\, {}^tSMS$ for some $S \in \mathrm{GL}_d(K)$ and $\lambda \in K^{\times}$, where ${}^tS$ is the transpose of S. Studying symmetric determinantal representations is a classical topic in algebraic geometry and linear algebra, which goes back to Hesse's work on plane cubics and quartics; [4], [5, Ch 4]. Recently, arithmetic properties of symmetric determinantal representations and related linear orbits are studied by several people; [2,8–13].

In this paper, we prove the local–global principle for the existence of symmetric determinantal representations in characteristic two.

Theorem 1.1 *(see Theorem 4.1). Let K be a global field of* characteristic two, *and $C \subset \mathbb{P}^2$ a smooth plane curve of degree $d \geq 1$ over K. Then the following are equivalent:*

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1. *C admits a symmetric determinantal representation over K.*
2. *C admits a symmetric determinantal representation over K_v for every place v of K.*

In our previous paper [13], we studied the local–global principle for the existence of symmetric determinantal representations over a global field of characteristic $\neq 2$. We proved the local–global principle for conics and cubics; see [13, Theorem 1.1]. For quartics, we constructed examples failing the local–global principle under mild conditions on the characteristic of the global field; see [13, Corollary 1.3]. According to some group theoretic results in [13, Section 6], we expect that similar examples failing the local–global principle exist in any degree ≥ 4 in any characteristic $\neq 2$; see [13, Problem 1.6].

In this paper, we study the remaining case of *characteristic two*. Rather interestingly, the story is completely different. The local–global principle for the existence of symmetric determinantal representations holds in any degree in characteristic two. The key is Mumford's theory of *canonical theta characteristics* on smooth projective curves which exist only in characteristic two.

The outline of this paper is as follows. After recalling a relation between symmetric determinantal representations and non-effective theta characteristics in Section 2, we recall Mumford's theory of canonical theta characteristics in Section 3. The main theorem is proved in Section 4. Finally, in Section 5, we consider an analogous problem for *linear determinantal representations* (i.e. the matrix M is not assumed to be symmetric). It turns out that, concerning the existence of linear determinantal representations, the local–global principle does not hold even for cubics. Hence Theorem 1.1 cannot be generalized to the linear case.

1.1. Notation

An algebraic closure of a field K is denoted by $\overline{K}$, and a separable closure of K is denoted by K^{sep}. A global field of characteristic two is a finite extension of $\mathbb{F}_2(T)$, where $\mathbb{F}_2$ is the finite field with two elements and T is an indeterminate. For a place v of K, the completion of K at v is denoted by K_v.

2. Symmetric determinantal representations and theta characteristics

In this section, let K be an arbitrary field.

Definition 2.1. Let C be a projective smooth geometrically connected curve over K. A *theta characteristic* on C is a line bundle $\mathscr{L}$ satisfying $\mathscr{L} \otimes \mathscr{L} \cong \Omega^1_C$, where Ω^1_C is the canonical sheaf on C. A theta characteristic $\mathscr{L}$ on C is *effective* (resp. *non-effective*) if $H^0(C, \mathscr{L}) \neq 0$ (resp. $H^0(C, \mathscr{L}) = 0$).

The following result is classical and well-known at least when the base field is algebraically closed of characteristic zero; see [4], [5, Ch 4]. In fact, it is valid over arbitrary fields.

Proposition 2.2 *(Beauville). Let $C \subset \mathbb{P}^2$ be a smooth plane curve over K. There is a natural bijection between the following sets:*

- *the set of equivalence classes of symmetric determinantal representations of C over K, and*
- *the set of isomorphism classes of non-effective theta characteristics on C defined over K.*

Proof. See [1, Proposition 4.2]. See also [12, Proposition 2.2, Corollary 2.3], [13, Theorem 2.2]. □

3. Canonical theta characteristics in characteristic two

In this section, let K be a field of *characteristic two*.

In his foundational paper on theta characteristics, Mumford observed the following "strange" fact; see [16, p. 191]. Let C be a projective smooth geometrically connected curve over K, and $K(C)$ the function