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The analogue of Hilbert's 1888 theorem for even symmetric forms

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A B S T R A C T

Hilbert proved in 1888 that a positive semidefinite (psd) real form is a sum of squares (sos) of real forms if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$, where n is the number of variables and $2d$ the degree of the form. We study the analogue for even symmetric forms. We establish that an even symmetric n-ary $2d$-ic psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n\geq 3}$ or $(n, 2d) = (3, 8)$.

1. Introduction

A real form (homogeneous polynomial) f is called *positive semidefinite* (psd) if it takes only non-negative values and it is called a *sum of squares* (sos) if there exist other forms h_j so that $f = h_1^2 + \cdots + h_k^2$. Let $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ denote the cone of psd and sos n-ary $2d$-ic forms (i.e. forms of degree $2d$ in n variables) respectively.

In 1888, Hilbert [9] gave a celebrated theorem that characterizes the pairs $(n, 2d)$ for which every n-ary $2d$-ic psd form can be written as a sos of forms. It states that every n-ary $2d$-ic psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$. Hilbert demonstrated that $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for $(n, 2d) = (4, 4), (3, 6)$, thus reducing the problem to these two basic cases.

Almost ninety years later, Choi and Lam [1] returned to this subject. In particular, they considered the question of when a symmetric psd form is sos. A form $f(x_1, \dots, x_n)$ is called *symmetric* if $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$ for all $\sigma \in S_n$. As an analogue of Hilbert's approach, they reduced the problem to finding symmetric psd not sos n-ary $2d$-ics for the pairs $(n, 4)_{n\geq 4}$ and $(3, 6)$. They asserted the existence of psd not sos symmetric quartics in $n \geq 5$ variables; contingent on these examples, the answer is the same as that found by Hilbert. In [6], we constructed these quartic forms.

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A form is *even symmetric* if it is symmetric and in each of its terms every variable has even degree. Let $S\mathcal{P}^e_{n,2d}$ and $S\Sigma^e_{n,2d}$ denote the set of even symmetric psd and even symmetric sos n-ary $2d$-ic forms respectively. Set $\Delta_{n,2d} := S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$. In this paper, we investigate the following question:

$$\mathcal{Q}(S^e) : \text{For what pairs } (n, 2d) \text{ is } \Delta_{n,2d} = \emptyset \text{ ?}$$

The current answers to this question in the literature are $\Delta_{n,2d} = \emptyset$ if $n = 2, d = 1, (n, 2d) = (3, 4)$ by Hilbert's Theorem, $(n, 2d) = (3, 8)$ due to Harris [7], and $(n, 2d) = (n, 4)_{n\geq 4}$. The result $\Delta_{n,4} = \emptyset$ for $n \geq 4$ was attributed to Choi, Lam and Reznick in [7]; a proof can be found in [5, Proposition 4.1]. Further, $\Delta_{n,2d} \neq \emptyset$ for $(n, 2d) = (n, 6)_{n\geq 3}$ due to Choi, Lam and Reznick [3], for $(n, 2d) = (3, 10), (4, 8)$ due to Harris [8] and for $(n, 2d) = (3, 6)$ due to Robinson [10]. Robinson's even symmetric psd not sos ternary sextic is the form

$$R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2.$$

Thus the answer to $\mathcal{Q}(S^e)$ in the literature can be summarized by the following chart:

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	o	o
10	✓	×	o	o	o
12	✓	o	o	o	o
14	✓	o	o	o	o
⋮	⋮	o	o	o	o

where, a tick (✓) denotes a positive answer to $\mathcal{Q}(S^e)$, a cross (×) denotes a negative answer to $\mathcal{Q}(S^e)$, and a circle (o) denotes "undetermined". Indeed to get a complete answer to $\mathcal{Q}(S^e)$, we need to investigate the question in these remaining cases, namely $(n, 8)$ for $n \geq 5$, $(3, 2d)$ for $d \geq 6$ and $(n, 2d)$ for $n \geq 4, d \geq 5$.

Main Theorem. *An even symmetric n-ary $2d$-ic psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n\geq 3}$ or $(n, 2d) = (3, 8)$.*

In other words, every "o" in the chart can be replaced by "×".

The article is structured as follows. In Section 2, we develop the tools (Theorem 2.3 and Theorem 2.4) we need to prove our Main Theorem. These tools allow us to reduce to certain basic cases, in the same spirit as Hilbert and Choi–Lam. In Section 3 and Section 4 we resolve those basic cases by producing explicit examples for $(n, 2d)$; $n \geq 4$, $d = 4, 5, 6$. We conclude Section 4 by interpreting even symmetric psd forms in terms of preorderings using our Main Theorem. Finally, for ease of reference we summarize our examples in Section 5.

2. Reduction to basic cases

The following Lemma will be used in Theorem 2.3.

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