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A coherence theorem for pseudonatural transformations

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ABSTRACT

We prove coherence theorems for bicategories, pseudofunctors and pseudonatural transformations. These theorems boil down to proving the coherence of some free $(4,2)$-categories. In the case of bicategories and pseudofunctors, existing rewriting techniques based on Squier's Theorem allow us to conclude. In the case of pseudonatural transformations this approach only proves the coherence of part of the structure, and we use a new rewriting result to conclude. To this end, we introduce the notions of white-categories and partial coherence.

0. Introduction

0.1. An overview of coherence theorems

A mathematical structure, such as the notion of monoid or algebra, is often defined in terms of some data satisfying relations. In the case of monoids, the data is a set and a binary application, and the relations are the associativity and the unit axioms. In category theory, one often considers relations that only hold *up to isomorphism*. One of the simplest example of such a structure is that of monoidal categories, in which the product is not associative, but instead there exist isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$. This additional data must also satisfy some relation, known as Mac–Lane's pentagon:

$$\begin{array}{ccccc} & (A\otimes(B\otimes C))\otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} & A\otimes((B\otimes C)\otimes D) & \\ {}^{\alpha_{A,B,C}\otimes D}\nearrow & & & & \searrow^{A\otimes\alpha_{B,C,D}} \\ ((A\otimes B)\otimes C)\otimes D & & = & & A\otimes(B\otimes(C\otimes D)) \\ & {}_{\alpha_{A\otimes B,C,D}}\searrow & & \nearrow_{\alpha_{A,B,C\otimes D}} & \\ & & (A\otimes B)\otimes(C\otimes D) & & \end{array}$$

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The intended purpose of this relation is that, between any two bracketings of $A_1 \otimes A_2 \otimes \ldots \otimes A_{n-1} \otimes A_n$, there exists a unique isomorphism constructed from the isomorphisms $\alpha_{A,B,C}$. This statement was made precise and proved by Mac Lane in the case of monoidal categories [12]. In general a *coherence theorem* contains a description of a certain class of diagrams that are to commute. Coherence theorems exist for various other structures, e.g. bicategories [13], or V-natural transformations for a symmetric monoidal closed category V [10].

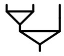
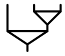
Coherence results are often a consequence of (arguably more essential [9]) *strictification theorems*. A strictification theorem states that a "weak" structure is equivalent to a "strict" (or at least "stricter") one. For example any bicategory is biequivalent to a 2-category, and the same is true for pseudofunctors (this is a consequence of this general strictification result [15]). It does not hold however for pseudonatural transformations.

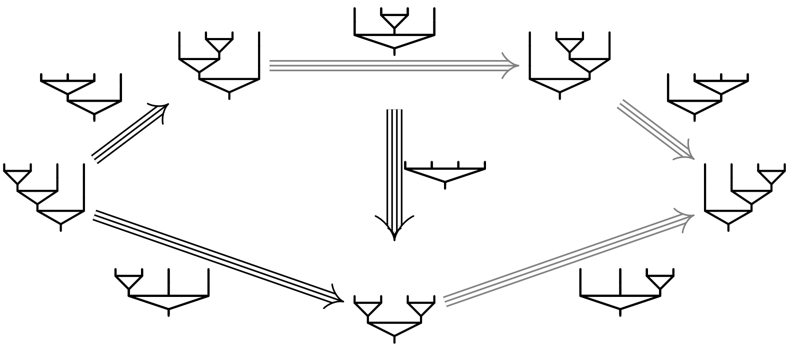
0.2. Free categories and rewriting

Coherence theorems can also be proven through rewriting techniques. The link between coherence and rewriting goes back to Squier's homotopical Theorem [16], and has since been expanded upon [6]. Squier's theory is constructive, which means that the coherence conditions can be calculated from the relations, in a potentially automatic way. It can also be expanded to higher dimensions [8], a feature that may prove useful when studying weaker structures. In [7], the authors use Squier's theory to prove the coherence of monoidal categories. Let us give an outline of the proof in the case of categories equipped with an associative tensor product.

Polygraphs are presentations for higher-dimensional categories and were introduced by Burroni [3], and by Street under the name of computads [17,18]. In this paper we use Burroni's terminology. For example, a 1-polygraph is given by a graph G, and the free 1-category it generates is the category of paths on G. If Σ is an n-polygraph, we denote by Σ^* the free n-category generated by Σ.

An (n,p)-*category* is a category where all k-cells are invertible, for $k > p$. In particular, $(n,0)$-categories are commonly called n-groupoids, and (n,n)-categories are just n-categories. There is a corresponding notion of (n,p)-polygraph. If Σ is an (n,p)-polygraph, we denote by $\Sigma^{*(p)}$ the free (n,p)-category generated by Σ.

The structure of category equipped with an associative tensor product is encoded into a 4-polygraph **Assoc**, which generates a free $(4,2)$-category $\mathbf{Assoc}^{*(2)}$. The 4-polygraph **Assoc** contains one generating 2-cell  coding for product, one generating 3-cell  :  $\Rrightarrow$  coding for associativity and one generating 4-cell  corresponding to Mac Lane's pentagon:



The coherence result for categories equipped with an associative product is now reduced to showing that, between every parallel 3-cells A, B in $\mathbf{Assoc}^{*(2)}$, there exists a 4-cell $\alpha : A \Rrightarrow B$ in $\mathbf{Assoc}^{*(2)}$. A 4-category satisfying this property is said to be 3-*coherent*.

Let us denote by $\mathbf{Assoc}^*$ the free 4-category generated by **Assoc**. We have the following properties:

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