Linear Algebra and its Applications 537 (2018) 22–37

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Quantity of operators with Bhatia–Šemrl property

Sun Kwang Kim [1]

Department of Mathematics, Kyonggi University, Suwon 16227, Republic of Korea

ARTICLE INFO

Article history:
Received 28 May 2017
Accepted 19 September 2017
Available online 21 September 2017
Submitted by R. Bhatia

MSC:
primary 46B20
secondary 46B04, 46B22, 46B2

Keywords:
Birkhoff–James orthogonality
Bhatia–Šemrl property
Banach space

ABSTRACT

We study the Bhatia–Šemrl property which is related to Birkhoff–James orthogonality on operator spaces. Recently, many authors obtained some conditions for operators to have Bhatia–Šemrl property. Using them, we explore the denseness of operators with Bhatia–Šemrl property or without Bhatia–Šemrl property on some classes of Banach spaces.

1. Introduction

In 1935, G. Birkhoff considered a notion of orthogonality in linear metric spaces which we call Birkhoff–James orthogonality [3]. This concept has been played a very important role for studying geometry of Banach space. There has been many interesting works about orthogonality on operator space. Among them, on finite dimensional Hilbert space, Bhatia and Šemrl [2] found necessary and sufficient condition for operators A and B to be that A is orthogonal to B in terms of specific vectors. They also asked whether the

E-mail address: sunkwang@kgu.ac.kr.

[1] This work was supported by Kyonggi University Research Grant 2016.

https://doi.org/10.1016/j.laa.2017.09.021

same characterization holds for arbitrary finite dimensional space, and later a negative answer was given by Li and Schneider [7]. In this paper, we investigate the condition of Bhatia and Šemrl. Especially, we focus on 'quantity of operators having the condition'.

Let us now restart the introduction. This time we give necessary notions and background material. Let X and Y be Banach spaces. We will use the common notation S_X, B_X, X^* for the unit sphere, the closed unit ball and the dual space of X respectively, $\mathcal{L}(X, Y)$ for the Banach space of every bounded linear operators from X to Y. For the convenience we write $\mathcal{L}(X)$ to denote $\mathcal{L}(X, X)$. We say that an operator $T \in \mathcal{L}(X, Y)$ attains its norm if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\| = \sup_{x \in B_X} \|T(x)\|$. We use M_T for the set of points where T attains its norm.

For any $x, y \in X$, x is said to be orthogonal to y in the sense of Birkhoff–James ($x \perp_B y$) if $\|x\| \leq \|x + \lambda y\|$ for every $\lambda \in \mathbb{R}$ [3]. On the operator space $\mathcal{L}(X, Y)$, it is clear that for T and A in $\mathcal{L}(X, Y)$ if there exists $x \in S_X$ such that $\|T(x)\| = \|T\|$ and $T(x) \perp_B A(x)$, then $T \perp_B A$. When H is finite dimensional Hilbert space, the converse for $\mathcal{L}(H)$ was proved by Bhatia and Šemrl [2]. Indeed, for a finite dimensional Hilbert space H and $T, A \in \mathcal{H}$,

$$T \perp_B A \Leftrightarrow \exists x \in M_T \text{ such that } Tx \perp_B Ax.$$

Later, this is generalized by Benítez, Fernández and Soriano to inner product space [1].

Motivated by above results, Bhatia–Šemrl property is defined as follows.

Definition 1.1. [11] For Banach spaces X and Y, an operator $T \in \mathcal{L}(X, Y)$ said to have Bhatia–Šemrl property (BŠ property) if for each A with $T \perp_B A$ there exists $x \in M_T$ such that $Tx \perp_B Ax$.

After it is known that there exists some finite dimensional normed space X and an operator $T \in \mathcal{L}(X)$ without BŠ property [7], many authors tried find sufficient condition for T to have BŠ property (see [8–11]). Especially, Sain, Paul and Hait [11] proved that there are sufficiently many operators with BŠ property in $\mathcal{L}(X)$ when X is finite dimensional strictly convex real Banach space. More precisely, they showed that the set of operators having BŠ property is dense in $\mathcal{L}(X)$ for such X. Moreover, they remarked that strict convexity is not necessary by giving example which satisfies such situation.

Our aim in this paper is to get the generalization of the result of Sain, Paul and Hait. Especially, we deal with not only finite dimensional space but also some infinite dimensional Banach spaces, and just in case there exists an operator which does not attains its norm. Hence, we will consider only norm attaining operators for BŠ property in the remaining part of this paper.

In section 2, we show that for a Banach space X with Radon–Nikodým property, the set of norm attaining operators having BŠ property is dense in $\mathcal{L}(X, Y)$. It is worth to note that the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$ when X has Radon–Nikodým property due to a result of Bourgain [4]. There are many classes of Banach