Linear Algebra and its Applications 537 (2018) 48–83

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Eigenvalues of non-Hermitian random matrices and Brown measure of non-normal operators: Hermitian reduction and linearization method

Serban T. Belinschi [a,b], Piotr Śniady [c,*], Roland Speicher [d]

[a] CNRS, Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118, route de Narbonne, F-31062 Toulouse Cedex 9, France
[b] Department of Mathematics & Statistics, Queens University, Jeffery Hall, 48 University Avenue, Kingston, ON K7L 3N6, Canada
[c] Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland
[d] Universität des Saarlandes, Fachrichtung 6.1 – Mathematik, Postfach 151150, 66041 Saarbrücken, Germany

A R T I C L E   I N F O

Article history:
Received 21 March 2017
Accepted 20 September 2017
Available online 22 September 2017
Submitted by H. Rauhut

MSC:
primary 15B52
secondary 46L54, 46L10

Keywords:
Eigenvalues of non-Hermitian random matrices
Brown measure
Non-normal operators

A B S T R A C T

We study the Brown measure of certain non-Hermitian operators arising from Voiculescu's free probability theory. Usually those operators appear as the limit in $\star$-moments of certain ensembles of non-Hermitian random matrices, and the Brown measure gives then a canonical candidate for the limit eigenvalue distribution of the random matrices. A prominent class for our operators is given by polynomials in $\star$-free variables. Other explicit examples include R-diagonal elements and elliptic elements, for which the Brown measure was already known, and a new class of triangular-elliptic elements. Our method for the calculation of the Brown measure is based on a rigorous mathematical treatment of the Hermitian reduction method, as considered in the physical literature, combined with subordination and the linearization trick.

* Corresponding author.
E-mail addresses: serban.belinschi@math.univ-toulouse.fr (S.T. Belinschi), psniady@impan.pl (P. Śniady), speicher@math.uni-sb.de (R. Speicher).

https://doi.org/10.1016/j.laa.2017.09.024
0024-3795/

1. Introduction

1.1. Eigenvalues of non-Hermitian random matrices

The study of the eigenvalue distribution of non-Hermitian random matrices is regarded as an important and interesting problem, especially in the mathematical physics literature. Unfortunately, most of the methods used for the study of Hermitian random matrices fail in the non-Hermitian case, which makes the latter very difficult.

1.2. Convergence of $\star$-moments. Free probability theory

Usually we are interested in the behavior of the random matrix eigenvalues in the limit as the size of the matrix tends to infinity. It is therefore natural to ask: does a given sequence of random matrices converge in one sense or another to some (infinite-dimensional) object as the size of the matrix tends to infinity? It would be very tempting to study this limit instead of the sequence of random matrices itself.

In order to perform this program, we will use the notion of a $W^\star$-probability space (which is a von Neumann algebra $\mathfrak{A}$ equipped with a tracial, faithful, normal state $\phi : \mathfrak{A} \to \mathbb{C}$). The algebra $\mathfrak{A}_N = \mathcal{L}^{\infty-}(\Omega, \mathcal{M}_N)$ of $N \times N$ random matrices with all moments finite equipped with a tracial state $\phi_N(x) = \frac{1}{N}\mathbb{E}\operatorname{Tr} x$ for $x \in \mathfrak{A}_N$ fits well into this framework (to be very precise: the definition of a von Neumann algebra requires its elements to be bounded which is not the case for the most interesting examples of random matrices, but this small abuse of notation will not cause any problems in the following).

We say that a sequence of random matrices (A_N), where $A_N \in \mathfrak{A}_N$, converges in $\star$-moments to some element $x \in \mathfrak{A}$ if for every choice of $s_1, \dots, s_n \in \{1, \star\}$ we have

$$\lim_{N\to\infty} \phi_N(A_N^{s_1} \cdots A_N^{s_n}) = \phi(x^{s_1} \cdots x^{s_n}).$$

It turns out that many classes of random matrices have a limit in a sense of $\star$-moments and the limit operator can be found by the means of free probability theory [40,24,28,37].

1.3. Brown measure

The Brown measure [10] is an analogue of the density of eigenvalues for elements of $W^\star$-probability spaces. Its great advantage is that it is well-defined not only for self-adjoint or normal operators; furthermore for random matrices it coincides with the mean empirical eigenvalue distribution. We recall the exact definition in Section 2.1.

1.4. The main tools: (i) Hermitian reduction method

In this article we study rigorously the idea of Janik, Nowak, Papp and Zahed [26] which was later used in the papers [15,13,14] under the name *Hermitian reduction method*.