Linear Algebra and its Applications 537 (2018) 191–220

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A non-holomorphic functional calculus and the complex conjugate of a matrix

Olavi Nevanlinna

Aalto University, Department of Mathematics and Systems Analysis, Finland

A R T I C L E I N F O

Article history:
Received 7 February 2017
Accepted 4 October 2017
Available online 6 October 2017
Submitted by D.B. Szyld

MSC:
47A60
47A56
30E99

Keywords:
Functional calculus
Non-holomorphic
Nondiagonalizable matrices

A B S T R A C T

Based on Stokes' theorem we derive a non-holomorphic functional calculus for matrices, assuming sufficient smoothness near eigenvalues, corresponding to the size of related Jordan blocks. It is then applied to the complex conjugation function $\tau : z \mapsto \bar{z}$. The resulting matrix agrees with the hermitian transpose if and only if the matrix is normal. Two other, as such elementary, approaches to define the complex conjugate of a matrix yield the same result.

1. Overview

We discuss a simple non-holomorphic functional calculus for complex $n \times n$-matrices $A \in \mathbb{M}_n(\mathbb{C})$ with particular focus in defining $\tau(A) \in \mathbb{M}_n(\mathbb{C})$ for the complex conjugation function

$$\tau : z \mapsto \bar{z}. \tag{1.1}$$

E-mail address: Olavi.Nevanlinna@aalto.fi.

https://doi.org/10.1016/j.laa.2017.10.004

We shall denote the resulting matrix $\tau(A)$ by A^c and it agrees with A^* if and only if A is normal.

Theorem 1.19 in [18] says that if Ω is an open set either of $\mathbb{R}$ or $\mathbb{C}$, and if φ is $n-1$ times continuously differentiable on the set Ω, then $\varphi(A)$ is a continuous matrix function on the set of matrices $A \in \mathbb{M}_n(\mathbb{C})$ with spectrum in Ω. When Ω is open in $\mathbb{C}$, then φ is in fact assumed to be holomorphic. Our non-holomorphic calculus shall be well defined for all functions φ in $C^{n-1}(\Omega)$, meaning that φ has continuous partial derivatives up to oder $n-1$ with respect to x and y, viewed as independent variables.

Recall first some basic facts on defining $\varphi(A)$ when φ is holomorphic. For φ holomorphic in Ω and continuous up to the boundary, the Cauchy integral

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad z \in \Omega \tag{1.2}$$

yields the basic definition for $\varphi(A)$, if Ω contains the eigenvalues and the Cauchy kernel is replaced by the resolvent

$$\zeta \mapsto (\zeta I - A)^{-1}. \tag{1.3}$$

The integral can be evaluated by computing residues at the eigenvalues. (On the history of definitions of $\varphi(A)$, see e.g. Section 1.10 in [18], in particular Frobenius [14] used residues to define matrix functions.) Observe that the use of Cauchy integral does not require transforming the matrix into Jordan canonical form and representations for the resolvent can be available along the contour, without need to know the exact locations of eigenvalues.

Assume now that $\varphi \in C^1(\overline{\Omega})$. The Cauchy integral naturally still defines a holomorphic function in Ω but if φ is not holomorphic, an area integral is needed:

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\varphi(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\Omega} \frac{\overline{\partial}\varphi(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}, \quad z \in \Omega. \tag{1.4}$$

This is often referred to as Cauchy–Green or Pompeiu's formula, [1,17]. Here $\overline{\partial}$ denotes the differential operator $\frac{1}{2}(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta})$ when $\zeta = \xi + i\eta$ is the complex variable. Notice that φ is holomorphic if and only if $\overline{\partial}\varphi$ vanishes in Ω. Again one can try to replace the kernel by the resolvent and in fact such approaches exist, with special restrictions on φ and on the operator to guarantee the convergence of the area integral, [5,9–11,16,20]. We focus here on matrices and therefore all singularities are poles, but in the presence of nontrivial Jordan blocks higher order poles show up and these are not covered by (1.4). However, for diagonalizable matrices $A = TDT^{-1}$ the formula yields immediately

$$\frac{1}{2\pi i} \int_{\partial\Omega} \varphi(\zeta)(\zeta I - A)^{-1} d\zeta + \frac{1}{2\pi i} \int_{\Omega} \overline{\partial}\varphi(\zeta)(\zeta I - A)^{-1} d\zeta \wedge d\overline{\zeta} = T\varphi(D)T^{-1} \tag{1.5}$$