Linear Algebra and its Applications 537 (2018) 302–317

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The ray nonsingularity of certain uniformly random ray patterns ☆

Yue Liu

College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 350116, China

ARTICLE INFO

Article history:
Received 19 June 2017
Accepted 2 October 2017
Available online 7 October 2017
Submitted by R. Brualdi

MSC:
15A06
15A04

Keywords:
Ray pattern
Ray nonsingular
Uniform distribution
Cycle tree matrix
Digraph

ABSTRACT

A uniformly random ray pattern matrix A with a given zero-nonzero pattern (described by a digraph D with no multi-arcs or loops) is the matrix whose nonzero entries are mutually independent random variables uniformly distributed over the unit circle $\mathbf{S}^1$ in the complex plane. It is shown in this paper that the probability of $I - A$ to be ray nonsingular is completely determined by the cycle graph $\mathscr{CG}(D)$ of D (i.e. the adjacency structure of the directed cycles in D) if $\mathscr{CG}(D)$ is a tree. A formula is given to compute the probability when $\mathscr{CG}(D)$ is a tree, and it is also shown that as the order of $\mathscr{CG}(D)$ tends to infinity, the limit of the probability is 0.

1. Introduction

A complex matrix is called a *ray pattern matrix* if the modulus of its nonzero entries are all 1. The *ray pattern* of a complex matrix A, denoted by ray(A), is the ray pattern matrix obtained from A by replacing each nonzero entry a_{jk} by $a_{jk}/|a_{jk}|$. The set of complex matrices with the same ray pattern as A is called the *ray pattern class* of A, denoted by $Q(A)$, i.e.,

☆ This research is supported by National Natural Science Foundation of China 11471077 and 11571075.
E-mail address: liuyue@fzu.edu.cn.

https://doi.org/10.1016/j.laa.2017.10.003

$$Q(A) = \{B|\, \mathrm{ray}(B) = \mathrm{ray}(A)\}.$$

As a natural generalization of sign pattern matrices (see [1,2]) from the real case to the complex case, the study of ray pattern matrices has received considerable attentions during the last two decades. For example, in [3–5] and related articles, the spectrally arbitrary ray patterns are studied; in [6–8] the powers of ray pattern matrices are studied; in [9] the ray solvability of complex linear system is studied, and the Moore–Penrose inverse of ray pattern matrices is studied in [10].

A complex square matrix A is called *ray nonsingular* if every matrix B in the ray pattern class $Q(A)$ is nonsingular. In other words, ray nonsingular matrices (abbreviated as *RNS matrices*) are the matrices that can be confirmed to be nonsingular when only the ray patterns of the matrices are provided. By the definition, it is easy to see that whether or not A is RNS is completely determined by its ray pattern $\mathrm{ray}(A)$. Throughout this paper, we will only talk about complex square matrices and concern whether or not their ray patterns are RNS matrices.

RNS matrices are a generalization of the well known SNS matrices, which is a fundamental concept in the study of sign solvable linear systems (see [1] for detail). The concept was introduced together with the introduction of ray patterns in [11].

A natural problem concerning RNS matrices is:

Problem 1. Given a *randomly* selected ray pattern matrix A, what is the probability that A is ray nonsingular?

To discuss this problem rigorously, in Section 2 the definition of random variables uniformly distributed over the set of modulus 1 complex numbers is introduced. A uniformly random ray pattern matrix A with a given zero-nonzero pattern is defined to be the matrix whose nonzero entries are such kind mutually independent random variables, and the zero-nonzero pattern is described by a digraph D with no multi-arcs or loops.

A necessary condition for a ray pattern matrix M to be ray nonsingular is that M is of full term rank. So after proper permutations and ray scalings of the rows, just like when charactering SNS matrices we can assume the diagonal entries of the matrices to be all negative (see [1, Page 43] for example), without loss of generality, we may always require that $M = I - A$, where I is the identity matrix and A has zero diagonal. The probability for the uniformly random ray pattern matrix of the form $I - A$ to be ray nonsingular is studied in Section 3, a formula is given for the case when the adjacency structure of the directed cycles in D is a tree. It is also shown that as the order of this tree tends to infinity, the limit of the probability is 0.

2. The probability model of ray pattern matrices

Let $\mathbf{S}^1$ be the set of all the complex numbers with modulus 1, i.e., the unit circle in the complex plane. For two real numbers α and β satisfying $\beta - 2\pi \le \alpha \le \beta \le \alpha + 2\pi$, write